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A Nonlinear System Under Combined Periodic and Random Excitation

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The anharmonic oscillator under combined sinusoidal and white noise excitation is studied using the Gaussian closure approximation. The mean response and the steady-state variance of the system is obtained by the WKBJ approximation and also by the Fokker–Planck equation. The multiple steady-state solutions are obtained and their stability analysis is presented. Numerical results are obtained for a particular set of system parameters. The theoretical results are compared with a digital simulation study to bring out the usefulness of the present approximate theory.

KEY WORDS: Nonlinear equation; stochastic process; stability; steady-state; Gaussian closure.

1. INTRODUCTION

Duffing's equation (also called the anharmonic oscillator) has attracted much attention as a typical nonlinear system. Under purely sinusoidal excitations the system is known to possess multiple steady-state solutions. On the other hand, under a zero mean stationary random excitation the response is also a zero mean stationary random process. In the first case it is common to use the technique of averaging or harmonic linearization over one period of the solution. In the latter, random case the statistical linearization in the sense of ensemble averaging is popular. When the input is a combination of the two, it is natural to pursue a combination of the two types of linearizations for getting a solution. This is the approach taken by Caughey,(1) Budgor,(2) and Bulsara, Lindenberg and Schuler.(3) In the present paper the above nonlinear problem is studied by the Gaussian

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closure technique. This technique, previously described by Iyengar and Dash,[4] presupposes that certain joint probability density functions are Gaussian, to arrive at a closed hierarchy of moment equations through ensemble averaging. It is shown that the solution contains a periodic mean part and a random part, which attains stationarity in the long run. However, such a solution is found to be stable only for some values of the excitation frequency. Digital simulations have also been undertaken to a limited extent to verify the theoretical predictions.

2. NONLINEAR SYSTEM

The anharmonic oscillator is governed by the equation

$$\ddot{z} + 2\eta\omega\dot{z} + \omega^2 z + \beta z^3 = Q\lambda^2 \sin \lambda t + f(t) \tag{1}$$

where $f(t)$ is a Gaussian white noise process with autocorrelation

$$\langle f(t_1)\, f(t_2)\rangle = I\,\delta(t_2 - t_1) \tag{2}$$

and η is the viscous damping coefficient less than unity. Transformation of the response variable as

$$x = z/\sigma_1, \qquad \sigma_1^2 = I/(4\eta\omega^3) \tag{3}$$

leads to

$$\ddot{x} + 2\eta\omega\dot{x} + \omega^2 x + \beta\sigma_1^2 x^3 = (f/\sigma_1) + (\lambda^2 Q/\sigma_1) \sin \lambda t \tag{4}$$

Here, it may be noted that σ^2 is the steady-state variance of eq. (1) when $\beta = 0$. In this case the system is linear and hence the sinusoidal excitation contributes only to the mean response. However, in the nonlinear case this is no longer true. Also, when $f = 0$, the nonlinear oscillator exhibits multiple steady-state solutions. It is interesting to study what happens to these if the excitation contains a random part also.

3. GAUSSIAN CLOSURE

The response process is expressed as

$$x(t) = m(t) + y(t) \tag{5}$$

where m is the mean part and y is the random process part. The Gaussian closure approximation assumes that the random processes $y(t)$ and $f(t)$ are jointly normally distributed. Substitution of eq. (5) in eq. (4) gives

$$\begin{aligned}&[\ddot{m} + 2\eta\omega\dot{m} + \omega^2 m + \beta\sigma_1^2(m^3 + 3y^2 m)]\\ &\quad + [\ddot{y} + 2\eta\omega\dot{y} + \omega^2 z + \beta\sigma_1^2(y^3 + 3m^2 y)] = (f/\sigma_1) + (\lambda^2 Q/\sigma_1) \sin \lambda t \end{aligned} \tag{6}$$

Taking ensemble averages, the Gaussian assumption on y leads to

$$\ddot{m} + 2\eta\omega\dot{m} + \omega^2 m + \beta\sigma_1^2(m^3 + 3\sigma^2 m) = (\lambda^2 Q/\sigma_1)\sin\lambda t \tag{7}$$

Here, $\sigma(t)$ is the unknown standard deviation of the process $y(t)$. Now, multiplying eq. (6) by $f(t_1)$ and taking averages, one gets, for the cross correlation between $y(t)$ and $f(t_1)$, the equation

$$\ddot{R}_{yf} + 2\eta\omega\dot{R}_{yf} + \omega^2 R_{yf} + 3\beta\sigma_1^2(m^2 + \sigma^2)\, R_{yf} = R_{ff}(t, t_1)/\sigma_1 \tag{8}$$

Similarly one can derive an equation for the autocorrelation function $R_{yy}(t, t_1)$ also. These equations together are equivalent to the linear equation

$$\ddot{y} + 2\eta\omega\dot{y} + \omega^2 y + 3\beta\sigma_1^2(m^2 + \sigma^2)\, y = f/\sigma_1 \tag{9}$$

which has $m(t)$ and $\sigma(t)$ as time-varying coefficients. The solution of eq. (7) in the steady-state to the first approximation can be taken as

$$m = R\sin(\lambda t - \phi) \tag{10}$$

In case $y(t)$ attains stationarity, σ^2 will be a slowly varying function in comparison with $m(t)$ and will approach a constant value for large t. Thus, in the first approximation, σ may be treated as a constant in eqs. (7) and (9). From harmonic balance, eq. (7) may be analyzed to get

$$R^2 = \bar{Q}^2(\lambda/\omega)^4/[(1 + 3\varepsilon\sigma^2 + 0.75\varepsilon R^2 - \lambda^2/\omega^2)^2 + (2\eta\lambda/\omega)^2] \tag{11}$$

$$\tan\phi = 2\eta(\lambda/\omega)/[1 + 3\varepsilon\sigma^2 + 0.75\varepsilon R^2 - \lambda^2/\omega^2] \tag{12}$$

$$\bar{Q} = Q/\sigma_1 \tag{13}$$

$$\varepsilon = \beta\sigma_1^2/\omega^2 \tag{14}$$

It remains to solve eq. (9) along with the above equations. Since m is not a constant, eq. (9) cannot be solved exactly. Approximations via the WKBJ approach or the Fokker–Planck equation are, however, possible.

4. WKBJ APPROXIMATION

When ε is small, the solution of eq. (9) can be explicitly written as

$$y(t) = (1/\sigma_1)\int_0^t f(\tau)[g(t)\, g(\tau)^{-1/2}\exp[-\eta\omega(t-\tau)] \times \sin[g_1(t) - g_1(\tau)]\, d\tau \tag{15}$$

Here

$$g(t) = \mu[1 - \delta \cos 2(\lambda t - \phi)]^{1/2} \tag{16}$$

$$\mu^2 = \omega^2[1 - \eta^2 + 3\varepsilon(\sigma^2 + 0.5R^2)] \tag{17}$$

$$\delta = 1.5\varepsilon^2 R^2/\mu^2 \tag{18}$$

$$g_1(t) = \int g(t)\, dt \tag{19}$$

Since $f(t)$ is a white noise, it follows from standard random vibration theory, that

$$\sigma^2(t) = [I/g(t)\, \sigma_1^2] \int_0^t g^{-1}(\tau) \exp[-2\eta\omega(t-\tau)] \times \sin^2[g_1(t) - g_1(\tau)]\, d\tau \tag{20}$$

When the steady state is of primary interest, functions with $(t-\tau)$ as the argument will be the main contributing terms in the above integral. This essentially amounts to expanding the integrand and retaining the first few dominant terms. Thus it may be shown that

$$[g(t)\, g(\tau)]^{-1} = \mu^2[a_1^2 + 0.5a_2^2 \cos 2\lambda(t-\tau) + \cdots] \tag{21}$$

$$\cos 2[g_1(t) - g_1(\tau)] = \cos 2\mu a_3(t-\tau)[J_0^2(a_4) + 2J_1^2(a_4)\cos 2\lambda(t-\tau) + 2J_2^2(a_4)\cos 4\lambda(t-\tau) + \cdots] \tag{22}$$

$$a_1 = (1 + 3\,\delta^2/16) \tag{23}$$

$$a_2 = (0.5\delta + 15\delta^3/64) \tag{24}$$

$$a_3 = (1 - \delta^2/16) \tag{25}$$

$$a_4 = 0.5\mu\,\delta/\lambda \tag{26}$$

Here, $J_n(a_4)$ is the Bessel function of order n and argument a_4. After some more algebra it may be found that, with $\bar{\lambda} = \lambda/\omega$, $\bar{\mu} = \mu/\omega$

$$\sigma^2 = (\eta/\mu^2)((a_1/\eta) + [0.5a_2^2\bar{\lambda}/(\eta^2 + \bar{\lambda}^2)] - [b_1\bar{\mu}a_3/(\eta^2 + a_3^2\bar{\mu}^2)] - 0.5b_2[(\bar{\lambda} + a_3\bar{\mu})/[\eta^2 + (\bar{\lambda} + \bar{\mu}a_4)^2] + (\bar{\lambda} - a_3\bar{\mu})/[\eta^2 + (\bar{\lambda} - \bar{\mu}a_4)^2]\} - 0.5b_3\{(2\bar{\lambda} + a_3\bar{\mu})/[\eta^2 + (2\bar{\lambda} + \bar{\mu}a_4)^2] + (2\bar{\lambda} - a_3\bar{\mu})/[\eta^2 + (2\bar{\lambda} - \bar{\mu}a_4)^2]\}) \tag{27}$$

$$b_1 = (a_1 J_0^2 + 0.5 J_1^2 a_2^2) \tag{28}$$

$$b_2 = (2a_1^2 J_1^2 + 0.5a_2^2 J_0^2 + 0.5a_2^2 J_2^2) \tag{29}$$

$$b_3 = (2a_1^2 J_2^2 + 0.5a_2^2 J_1^2 + 0.5a_2^2 J_3^2) \tag{30}$$

Equations (11), (12), and (27) have to be solved simultaneously to find R, ϕ, and σ.

5. FOKKER–PLANCK EQUATION

An alternate to the above approximation, particularly attractive when $f(t)$ is white noise, is to write down the Fokker–Planck equation corresponding to eq. (9) and the joint density function $p(y, \dot{y}; t)$. This is easily obtained as

$$\frac{\partial p}{\partial t} = -\dot{y}\frac{\partial p}{\partial y} + \frac{\partial}{\partial \dot{y}}\{p[2\eta\omega\dot{y} + \omega^2 y + 3\beta\sigma_1^2(\sigma^2 + m^2)\, y]\} + 0.5I\sigma_1^2\frac{\partial^2 p}{\partial \dot{y}^2} \tag{31}$$

From this the equations for the second-order moments of y and $\dot{y}$, namely

$$s_1 = \langle y^2(t)\rangle = \sigma^2, \qquad s_2 = \langle \dot{y}^2(t)\rangle, \qquad s_3 = \langle y(t)\, \dot{y}(t)\rangle \tag{32}$$

are obtained as

$$ds_1/dt = 2s_3 \tag{33}$$

$$ds_2/dt = I/\sigma_1^2 - 2[\omega^2 + 3\beta\sigma_1^2(m^2 + \sigma^2)]\, s_3 - 4\eta\omega s_2 \tag{34}$$

$$ds_3/dt = s_2 - [\omega^2 + 3\beta\sigma_1^2(m^2 + \sigma^2)]\, s_1 - 2\eta\omega s_3 \tag{35}$$

When the steady state is of primary interest the time varying term $m^2(t)$ can be averaged over a period of oscillation to get approximately $\dot{m}^2 \simeq 0.5R^2$. Further in the steady state, the moment derivatives in eqs. (33–35) vanish leading to

$$\sigma^2 = \{[(1 + 1.5\varepsilon R^2)^2 + 12\varepsilon]^{1/2} - (1 + 1.5\varepsilon R^2)\}/(6\varepsilon) \tag{36}$$

Again, eqs. (11), (12), and (36) are to be solved simultaneously to arrive at numerical results on R, ϕ, and σ.

6. STABILITY ANALYSIS

In the absence of the random excitation $f(t)$, the amplitude R can have three solutions of which one is unstable. Thus it is possible that in the present case also R and hence, in turn σ, may exhibit three solutions.

However, these will be realizable only if they are stable also. This calls for the stability analysis of eq. (4), which is complicated due to the presence of the stochastic term. An approximate analysis is possible along the following lines. The sample solution of eq. (4) is in the form

$$x_0(t) = R\sin(\lambda t - \phi) + a\sin(\omega_e t - \theta) \tag{37}$$

where $a(t)$ and $\theta(t)$ are the slowly varying envelope and phase of the narrowband approximately ergodic Gaussian process $y(t)$. The dominant frequency present in $y(t)$ is the effective natural frequency of the nonlinear oscillator given by

$$\omega_e = \omega[1 + 3\varepsilon(0.5R^2 + \sigma^2)]^{1/2} \tag{38}$$

The above solution will be stable, provided small departures from this eventually vanish. This amounts to ascertaining the almost sure asymptotic stability of the variational equation of eq. (4) which is

$$\ddot{v} + 2\eta\omega\dot{v} + \omega^2 v + 3\beta\sigma_1^2 x_0^2 v = 0 \tag{39}$$

Now, introducing a nondimensional time $\omega t = \tau$ and with the transformation

$$v = ue^{-\eta\tau} \tag{40}$$

one gets

$$\begin{gathered} u'' + (C_0 - C_1\cos 2(\lambda_e\tau - \phi) - C_2\cos 2(\tau - \theta) \\ + C_3\{\cos[(1-\lambda_e)\tau - \theta + \phi] - \cos[(1+\lambda_e)\tau - \theta - \phi]\})\,u = 0 \\ \lambda_e = \lambda/\omega_e; \quad C_0 = 1 - \eta^2 + 1.5\varepsilon(a^2 + R^2) \\ C_1 = 1.5\varepsilon R^2; \quad C_2 = 1.5\varepsilon a^2; \quad C_3 = 3\varepsilon aR \end{gathered} \tag{41}$$

Here the primes denote derivatives with respect to τ. This equation contains the slowly varying stochastic coefficients a and θ and also the parametric frequencies 2, $2\lambda_e$, $(\lambda_e - 1)$, and $(\lambda_e + 1)$. In the primary harmonic region the dominant parametric frequency is 2, and hence one can take the solution for the above equation as

$$u = A\cos\tau + B\sin\tau \tag{42}$$

Since a and θ are slowly varying in comparison with the frequency of u, following the quasistatic approach of Stratonovich,[5] one can get averaged equations for A and B as

$$A' = -C_{11}A - C_{12}B; \qquad B' = C_{21}A + C_{22}B \tag{43}$$

The coefficients are

$$C_{11} = 0.25C_1 \sin 2\phi + [\cos 2\theta(1 - \cos 4\pi/\lambda_e)/(8\pi\omega_1)]\, C_2 - 0.5C_3 I_1$$

$$C_{12} = 0.5(\lambda_e^2 - C_0) - 0.25C_1 \cos 2\phi - 0.25C_2 \cos 2\theta + C_3 I_2$$

$$C_{21} = 0.5(\lambda_e^2 - C_0) + 0.25C_1 \cos 2\phi + 0.25C_2 \cos 2\theta - C_3 I_3$$

$$C_{22} = 0.25C_1 \sin 2\phi + [\cos 2\theta(1 - \cos 4\pi/\lambda_e)/(8\pi\omega_1)]\, C_2 + 0.5C_3 I_1$$

$$\omega_1 = (1 + \lambda_e^{-1}) \tag{44}$$

$$I_1 = (2\pi)^{-1} \int_0^{2\pi} \sin 2\psi[\cos(\psi/\lambda_e - \psi - \theta + \phi) - \cos(\psi/\lambda_e + \psi - \theta - \phi)]\, d\psi$$

$$I_2 = (2\pi)^{-1} \int_0^{2\pi} \sin^2 \psi[\cos(\psi/\lambda_e - \psi - \theta + \phi) - \cos(\psi/\lambda_e + \psi - \theta - \phi)]\, d\psi$$

$$I_3 = (2\pi)^{-1} \int_0^{2\pi} \cos^2 \psi[\cos(\psi/\lambda_e - \psi - \theta + \phi) - \cos(\psi/\lambda_e + \psi - \theta - \phi)]\, d\psi$$

Equation (43) can be satisfied by a solution of the type

$$A = A_0 \exp\left(\int_0^\tau \xi\, ds\right); \qquad B = B_0 \exp\left(\int_0^\tau \xi\, ds\right) \tag{45}$$

This gives

$$\xi(\tau) = 0.5(C_{22} - C_{11}) \pm 0.5[(C_{22} - C_{11})^2 - 4(C_{12}C_{21} - C_{11}C_{22})]^{1/2} \tag{46}$$

For almost sure asymptotic stability of the solution given by eq. (37) the condition would be

$$\lim_{\tau \to \infty} \exp\left\{-\eta - \text{real}\left[(1/\tau) \int_0^\tau \xi(s)\, ds\right]\right\} = 0 \tag{47}$$

Since a and θ are ergodic processes in the steady-state solution, ξ will also be ergodic, and hence the above time average can be replaced by the ensemble average to get the condition for stability as

$$\eta > \text{real}\langle \xi(a, \theta) \rangle \tag{48}$$

The joint density function of a and θ is given by

$$p(a, \theta) = (2\pi)^{-1} (a/\sigma^2) \exp(-0.5a^2/\sigma^2) \tag{49}$$

Since it can be shown that $\langle (C_{22} - C_{11}) \rangle = 0$ the stability condition further reduces to

$$\eta > 0.5 \iint_R (C_{22} - C_{11})^2 - 4(C_{12}C_{21} - C_{11}C_{22})^{1/2} \times p(a, \theta)\, da\, d\theta \quad (50)$$

where the integration is done over the region in which the integrand is real.

7. NUMERICAL EXAMPLE

Numerical results have been obtained from the above theory for a system with

$$\eta = 0.08, \qquad \varepsilon = 0.5, \qquad \bar{Q} = 0.5$$

The amplitude R of the mean solution and the corresponding variance σ^2 are shown in Figs. 1 and 2 as functions of λ/ω.

In Fig. 1 the deterministic response amplitude in the absence of the random excitation (i.e., $f = 0$) is also shown. The WKBJ solution and the

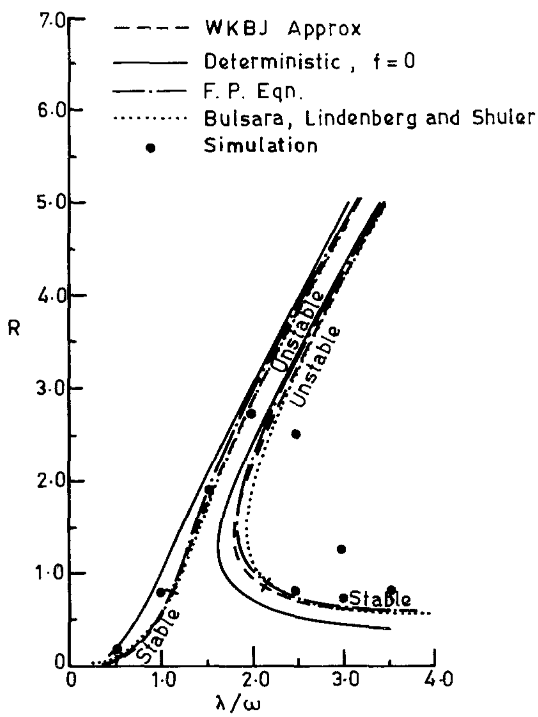


Fig. 1. Mean amplitude in steady-state for a system with $\varepsilon = 0.5$, $\eta = 0.08$, $\bar{Q} = 0.5$.

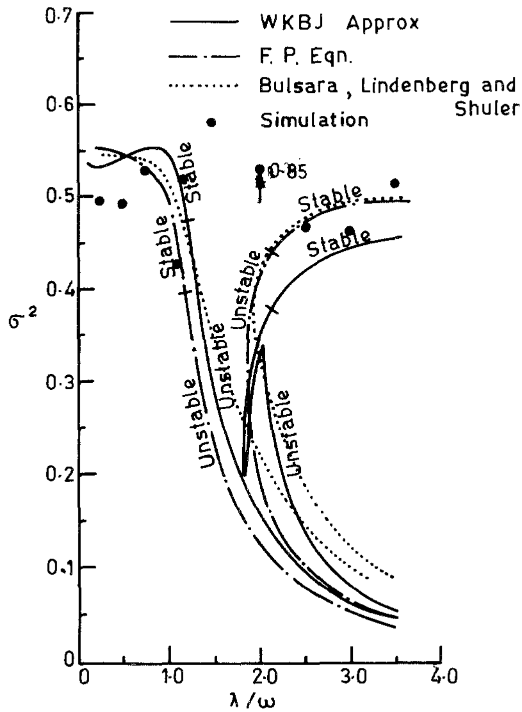


Fig. 2. Variance in steady-state for a system with $\varepsilon = 0.5$, $\eta = 0.08$, $\bar{Q} = 0.5$.

F–P equation solution for R are almost the same and are very similar to the deterministic case. The steady-state variance in the absence of nonlinearity (i.e., $\varepsilon = 0$) is unity. At the other extreme, in the absence of the random excitation, the variance is zero. The present steady-state value of variance must lie between zero and unity. Figure 2 shows the reduction of σ^2 from unity as λ/ω varies. Both the WKBJ and the F–P equation approaches give comparably similar results for σ^2. Simultaneously, the stability analysis has been carried out at every value of λ/ω to ascertain the realizability of the steady-state results. The stable and unstable regions are marked in Figs. 1 and 2. Bulsara, Lindenberg, and Shuler[3] have studied the present problem by a combination of harmonic and statistical averaging methods. It would be interesting to compare their results with the ones obtained here. In the notation of the present paper, the results of the above authors for R and σ^2 are

$$R = \bar{Q}\bar{\lambda}^2[4\eta^2\bar{\lambda}^2 + (\sigma^{-2} - \bar{\lambda}^2)^2]^{-1/2} \tag{51}$$

$$\sigma^{-2} = 1 + 3\varepsilon[\sigma^2 + (\bar{Q}^2\bar{\lambda}^4/D) + 0.125\bar{Q}^4\bar{\lambda}^8/(\sigma^2 D^2)] \times [1 + 0.5\bar{Q}^2\bar{\lambda}^4/(\sigma^2 D)]^{-1}$$
$$D = [4\eta^2\bar{\lambda}^2 + (\sigma^{-2} - \bar{\lambda}^2)^2] \tag{52}$$

These two equations have also been solved simultaneously and the results are shown in Figs. 1 and 2. It is seen that the present solution and the solution of Bulsara, Lindenberg, and Shuler[3] in general compare well.

8. NUMERICAL SIMULATION

In the absence of exact solutions, the above approximate theoretical predictions can be verified further through a numerical simulation of the basic eq. (4). This has been done by solving eq. (4) by the Runge–Kutta scheme for 100 samples of the white noise input. It would be convenient to measure time, in the numerical integration scheme in terms of cycles of oscillations $\tau = \omega t/2\pi$. This transforms eq. (4) to

$$x'' + 4\eta\pi x' + 4\pi^2 x + 4\pi^2 \varepsilon x^3 = 4\pi^2(\omega^2\sigma_1)^{-1} f(2\pi\tau/\omega) + 4\pi^2\bar{\lambda}^2\bar{Q}\sin(2\pi\bar{\lambda}\tau) \tag{53}$$

where the primes denote derivatives with respect to τ. The first term on the right side of this equation is the white noise process measured in the new time τ. The strength of this process is

$$I' = 16\pi^4(\omega^2\sigma_1)^{-2}(I\omega/2\pi) = 32\eta\pi^3 \tag{54}$$

In the numerical solution, this white noise process is simulated as a sequence of independent Gaussian random variables with zero mean and variance I'. The mean and variance are found by ensemble-averaging at every time instant across the 100 samples. The length of integration is to be based on the time required for the solutions to reach a steady-state. In the present context the steady-state is to be viewed as a stable periodic solution for the mean $m(t)$ and also a constant variance value for the process $y(t)$. For a linear system under white noise excitation, the approach to the steady-state depends on how fast $e^{-4\eta\pi\tau}$ approaches zero. For $\eta = 0.08$ this is achieved in less than 10 cycles. Thus, after allowing for the nonlinearity, the length of integration is taken as 50 cycles. The amplitude of the last cycle of the mean response is taken as an estimate of R and shown in Fig. 1. Similarly the average of the sample variance in the last cycle is taken as an estimate of the steady variance and plotted in Fig. 2. The convergence of the simulated mean and variance with respect to the sample size has been checked. It is found that for values of $\bar{\lambda}$ where the theoretical results are stable, 50 samples produce statistically acceptable results. In Figs. 3 and 4, typical simulated results of $m(\tau)$ and $\sigma^2(\tau)$ with 50 samples are shown for $\lambda/\omega = 1$ and 3, respectively. However, in Figs. 1 and 2 the simulated steady values are for 100 samples. In the stable regions of these figures the theoretical results compare favorably well with the simulated statistics. In the unstable region the sample variance plotted in Fig. 2 is largely different from the theoretical steady-state values. It must be noted that while in the unstable regions of Fig. 2, $y(t)$ does not tend to be a stationary process, the Gaussian closure method will still be able to yield good approximations. To demonstrate this, the moment eqs. (33–35) and

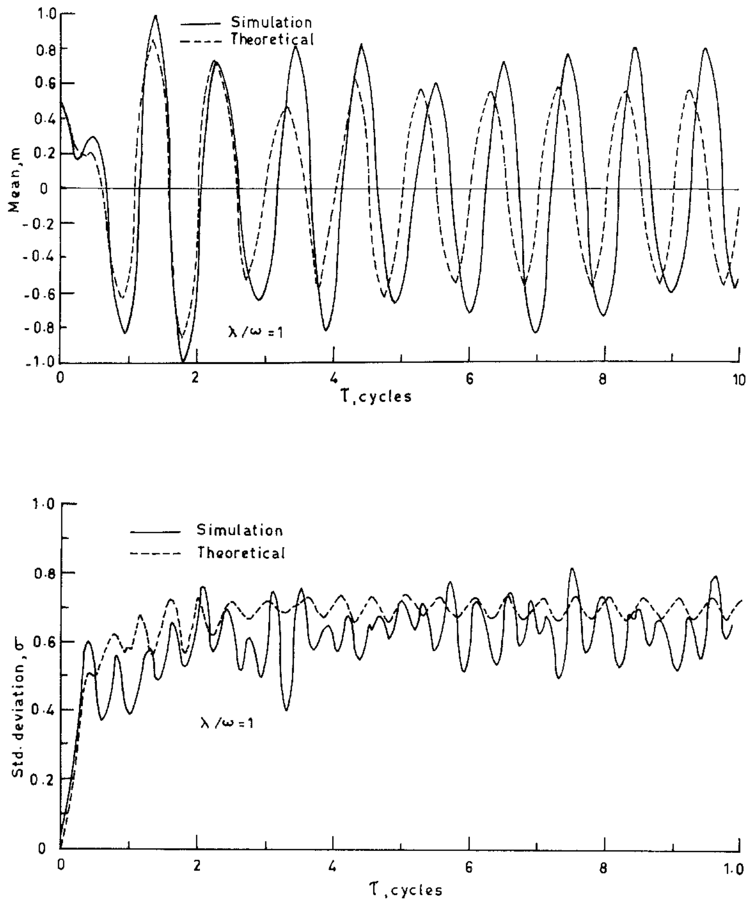
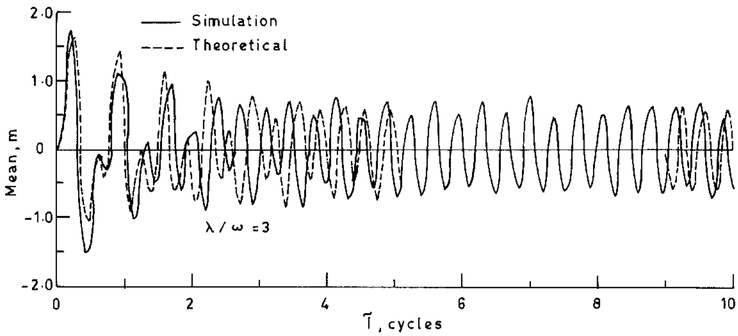


Fig. 3. (a) Nonstationary mean; $\lambda/\omega = 1$, (b) nonstationary standard deviation; $\lambda/\omega = 1$.



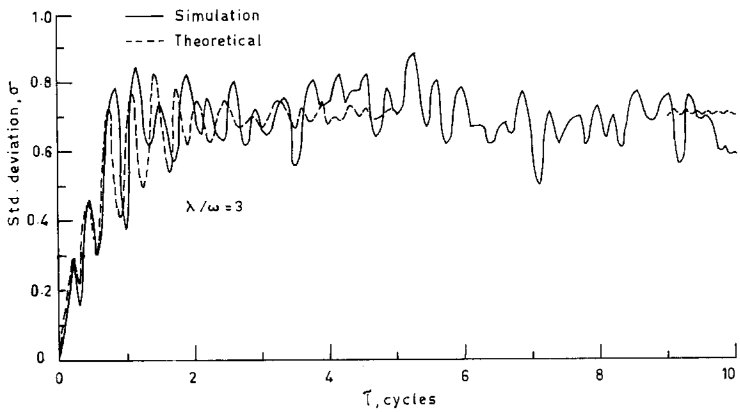


Fig. 4. (a) Nonstationary mean; $\lambda/\omega = 3$, (b) nonstationary standard deviation; $\lambda/\omega = 3$.

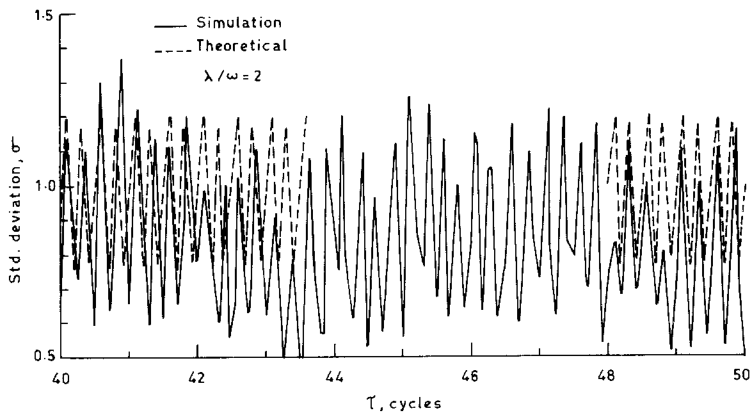


Fig. 5. Nonstationary standard deviation; $\lambda/\omega = 2$.

eq. (7) have been simultaneously solved numerically for $\lambda/\omega = 1$, 2, and 3 to obtain the time-dependent mean and variance without assuming a possible steady-state. These results are shown in Figs. 3, 4, and 5, along with the corresponding simulations on the exact equation. Again the present theory compares favorably with the numerical simulation.

9. DISCUSSION AND CONCLUSION

It is interesting to observe that introduction of a small random noise into a sinusoidal excitation can alter the response of a Duffing's oscillator considerably. The variance is here represented as a fraction of the steady-state variance of the linear case. It is known that for the hardening type of nonlinearity ($\varepsilon > 0$) the variance under white noise input ($\lambda = 0$) decreases fro the linear case. In the case of the combined excitation as in the present study, the mean and the second moments interact to reduce the variance further as λ, the frequency of the excitation increases. It would seem that as λ/ω_e approaches unity, the sinusoidal term drives the system lessening the randomness and thus increasing the mean but decreasing the response variance. On the other hand, away from resonance the white noise has considerable influence over the response and hence the variance increases with decreasing mean. However, the left branch of the solutions in Figs. 1 and 2 become unstable with increase in λ/ω and hence the steady-state itself may break down. For the example considered here, this happens at a value of about $\lambda/\omega = 1.2$. Beyond about $\lambda/\omega = 1.85$ three solutions become possible, but all of them are unstable. From about $\lambda/\omega = 2.1$ the smallest of the mean and correspondingly the largest variance becomes stable. The numerical simulations were carried out with the initial amplitude being near the stable deterministic value. Thus, for $\lambda/\omega = 2$, 2.5, 3, and 3.5 the simulation was repeated with two different initial conditions. For $\lambda/\omega = 2$, in both the cases instability was noticed. At $\lambda/\omega = 2.5$ and 3 the two different starting conditions lead to different mean values as shown, but to essentially the same variance value, at the end of 50 cycles.

The major result of this study is that the cubic oscillator of eq. (4) driven by both a noise and a harmonic term can have a steady-state solution that has a periodic mean and a nearly constant variance. However, the variance will cease to be constant for particular values of the external harmonic frequency. In conclusion, it may be noted here that the Gaussian closure technique provides a useful approach to study either the transient or stationary responses of nonlinear stochastic systems.

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REFERENCES

1. T. K. Caughey, *J. Acou. Soc. Am.* **35**:1706–1711 (1963).
2. A. B. Budgor, *J. Stat. Phys.* **17**:21–44 (1977).
3. A. R. Bulsara, K. Lindenberg, and K. E. Shuler, *J. Stat. Phys.* **27**:787–808 (1982).
4. R. N. Iyengar and P. K. Dash, *J. App. Mech.* **45**:393–399 (1978).
5. R. L. Stratonovich, *Topics in the Theory of Random Noise*, Vol. II (Gordon and Breach, New York, 1963).