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RANDOM VIBRATION OF A SECOND ORDER NON-LINEAR ELASTIC SYSTEM

R. NARAYANA IYENGAR

Department of Civil Engineering, Indian Institute of Science, Bangalore 560012, *India*

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A method is presented for obtaining, approximately, the response covariance and probability distribution of a non-linear oscillator under a Gaussian excitation. The method has similarities with the hierarchy closure and the equivalent linearization approaches, but is different. A Gaussianization technique is used to arrive at the output autocorrelation and the input–output cross-correlation. This along with an energy equivalence criterion is used to estimate the response distribution function. The method is applicable in both the transient and steady state response analysis under either stationary or non-stationary excitations. Good comparison has been observed between the predicted and the exact steady state probability distribution of a Duffing oscillator under a white noise input.

1. INTRODUCTION

The methods available at present for the analysis of non-linear systems under stochastic excitation are the Fokker–Planck equation approach [1, 2], the perturbation method [3, 4], the equivalent linearization technique [5, 6] and the hierarchy closure approximation [7, 8]. While the first method is useful only with white noise inputs, the others are more general. However, whereas the Fokker–Planck equation leads to the exact steady state distribution, the other methods stop generally with the response autocorrelations. Although, theoretically the perturbation method leads to non-Gaussian correction terms, finding the distribution of even the first of these is almost impossible. A typical result one would like to know is the joint probability density function of the response and its derivative, which is necessary in level crossing, peak and fatigue studies. At present, this result is known only for the steady state of systems under white noise inputs [1, 2]. For results in the transient state, the recent work of Atkinson [9] on the eigenfunction expansion method of solving the Fokker–Planck equation may be useful. For non-white inputs no general results on the level crossings or the peaks are available even for small non-linearities except a bound obtained by the author [10] using the worst input approach.

In the present study a linearly damped oscillator with cubic non-linearity is considered for detailed investigation. The response mean and autocorrelation are considered first. Deterministic differential equations are derived for these moments in terms of the known input moments. Even though the method is applicable to a wider class of systems the equation analyzed here represents the difficulties involved in non-linear random vibrations and also illustrates specifically the present approach. In the spirit of perturbation technique the method starts with that form of the probability distribution, of the input, response and response derivative, that one would obtain if the system were linear. Thus, for a non-linear system under a Gaussian excitation the natural starting point would be a Gaussian distribution. To arrive at the non-Gaussian response distribution, an analogous linear system is defined which has the same energy, at any time, as the original system. The energy equivalence relation leads to an estimate of the non-Gaussian displacement-velocity joint density function.

2. THE NON-LINEAR SYSTEM

The system considered for the analysis is described by the non-linear differential equation

$$\ddot{x} + 2\eta\omega\dot{x} + \omega^2 x + \alpha x^3 = f(t), \tag{1}$$

where η and ω are the fraction of critical damping and natural frequency, respectively, when the non-linear term is absent. $f(t)$ is a zero mean Gaussian process completely defined in terms of its autocorrelation function. Taking ensemble averages on both the sides of equation (1) yields

$$\frac{d^2\langle x\rangle}{dt^2} + 2\eta\omega\frac{d\langle x\rangle}{dt} + \omega^2\langle x\rangle + \alpha\langle x^3\rangle = \langle f(t)\rangle. \tag{2}$$

This illustrates the major difficulty in dealing with non-linear equations, namely the involvement of higher order moments in lower order moment equations. To overcome this another equation for $\langle x^3\rangle$ may be obtained, but this would lead to a hierarchy where still higher order moments are involved. It is precisely at this stage one invokes a closure assumption such as the cumulant discard approximation [7, 8]. Here, the troublesome higher order moments are handled directly by starting with the form of the joint density functions, $p_0(x_1,f_2;\, t_1,t_2)$ and $p_0(x_1,x_2;t_1,t_2)$, which one would get if the non-linearity was not present. Thus, for equation (1), since $f(t)$ is a Gaussian process,

$$p_0(x_1,t_2,t_1,t_2) = \frac{1}{2\pi\sigma_{x1}\sigma_{f2}\sqrt{1-r_{xf}^2}}\exp\left[-\frac{1}{2(1-r_{xf}^2)}\left\{\frac{x_1^2}{\sigma_{x1}^2} - \frac{2r_{xf}x_1f_2}{\sigma_{x1}\sigma_{f2}} + \frac{f_2^2}{\sigma_{f2}^2}\right\}\right], \tag{3}$$

where

$$\langle x^2(t_1)\rangle = \sigma_{x1}^2, \quad \langle f^2(t_2)\rangle = \sigma_{f2}^2,$$

$$\langle x(t_1)f(t_2)\rangle/\sigma_{x1}\sigma_{f2} = r_{xf}\,(t_1,t_2). \tag{4}$$

Also, the response itself is Gaussian with a two-dimensional density function

$$p_0(x_1,x_2,t_1,t_2) = \frac{1}{2\pi\sigma_{x1}\sigma_{x2}\sqrt{1-r_{xx}^2}}\exp\left[-\frac{1}{2(1-r_{xx}^2)}\left\{\frac{x_1^2}{\sigma_{x1}^2} - \frac{2r_{xx}x_1x_2}{\sigma_{x1}\sigma_{x2}} + \frac{x_2^2}{\sigma_{x2}}\right\}\right], \tag{5}$$

where

$$\langle x^2(t_2)\rangle = \sigma_{x2}^2, \quad \langle x(t_1)x(t_2)\rangle/\sigma_{x1}\sigma_{x2} = r_{xx}(t_1,t_2). \tag{6}$$

Hence it follows that

$$\langle x^3(t)\rangle = 0. \tag{7}$$

If the system starts from rest, that is,

$$x(0) = \dot{x}(0) = 0, \tag{8}$$

then equation (2) would give

$$\langle x(t)\rangle = 0. \tag{9}$$

2.1. THE CORRELATION FUNCTIONS

To obtain the autocorrelation function $R_{xx}(t,t_1) = \langle x(t)x(t_1)\rangle$, equation (1) is multiplied throughout by $x(t_1)$ and averaged. This yields

$$\ddot{R}_{xx}(t,t_1) + 2\eta\omega\dot{R}_{xx}(t,t_1) + \omega^2R_{xx}(t,t_1) + \alpha\langle x^3(t)x(t_1)\rangle = \langle x(t_1)f(t)\rangle = R_{xf}(t_1,t), \tag{10}$$

where the dots denote differentiation with respect to t and t_1 appears only as a parameter. But the right-hand side of this equation itself is an unknown. This difficulty is overcome by writing another equation for the input-output cross-correlation. By considering equation (1) with t_1

as the independent variable, multiplying by $f(t)$ throughout and averaging, a differential equation for $R_{xf}(t_1, t)$ can be obtained as

$$R''_{xf}(t_1, t) + 2\eta\omega R'_{xf}(t_1, t) + \omega^2 R_{xf}(t_1, t) + \alpha \langle x^3(t_1) f(t)\rangle = \langle f(t_1) f(t)\rangle = R_{ff}(t_1, t), \tag{11}$$

where the primes indicate differentiation with respect to t_1. This is an equation with t_1 as the independent variable and t as a parameter. Equation (10) has been derived previously by Crandall [11] in a similar context. But the approach has not been explored in detail.

Now, from the Gaussian initial approximations of equations (3) and (5) it follows that

$$\langle x^3(t_1) f(t)\rangle = 3\sigma_x^2(t_1) R_{xf}(t_1, t),$$

$$\langle x^3(t) x(t_1)\rangle = 3\sigma_x^2(t) R_{xx}(t, t_1). \tag{12}$$

These reduce equations (10) and (11) to the forms

$$R''_{xf}(t_1, t) + 2\eta\omega R'_{xf}(t_1, t) + [\omega^2 + 3\alpha\sigma_x^2(t_1)] R_{xf}(t_1, t) = R_{ff}(t_1, t), \tag{13}$$

$$\ddot{R}_{xx}(t, t_1) + 2\eta\omega \dot{R}_{xx}(t, t_1) + [\omega^2 + 3\alpha\sigma_x^2(t)] R_{xx}(t, t_1) = R_{xf}(t_1, t). \tag{14}$$

These look like coupled quasi-linear equations with time varying coefficients. The interesting feature is that the variance,

$$\sigma_x^2(t) = R_{xx}(t, t), \tag{15}$$

is *locked-in* as a time varying coefficient. The initial conditions are

$$R_{xf}(0, t) = R'_{xf}(0, t) = 0, \tag{16}$$

$$R_{xx}(0, t_1) = \dot{R}_{xx}(0, t_1) = 0. \tag{17}$$

A rigorous analysis of the above equations will not be attempted here. Although a closed form solution seems not possible in general, a numerical solution on a computer is quite feasible. A suitable simple function, ${}_1\sigma_x^2(t)$, is assumed for the output variance to start with. Then equation (13) is solved at various values of t with t_1 as the independent variable. This determines the right-hand side of equation (14) which can again be solved similarly to get the first approximation, ${}_1R_{xx}(t, t_1)$, to the response autocorrelation function. This would lead to the second approximation, ${}_2\sigma_x^2(t)$, for the output variance. This iterative process may be continued until a desired accuracy is attained. A plausible starting approximation would be the variance of the resulting linear system when $\alpha = 0$ in equation (1). However, herein, such a numerical work is not pursued; instead approximate solutions of equations (13) and (14) will be obtained for some specific random inputs.

3. STATIONARY EXCITATION

When the input is a stationary random process, in the presence of damping one would expect the existence of a steady state for large t and t_1 when the response also tends to a stationary process. This would lead to a constant variance, σ_s^2, for the response. With this in view the first approximation could be taken as

$${}_1\sigma_x^2(t) = \sigma_{ls}^2, \tag{18}$$

where σ_{ls}^2 is the known linear steady state variance. Substituting this in equations (13) and (14) one gets

$${}_1R_{xf}(t_1, t) = \frac{1}{\lambda_d} \int_0^t R_{ff}(\tau_1 - t)\, e^{-\zeta\lambda(t_1 - \tau_1)} \sin \lambda_d(t_1 - \tau_1)\, d\tau_1, \tag{19}$$

$$ {}_1R_{xx}(t, t_1) = \frac{1}{\lambda_d^2} \int_0^t \int_0^{t_1} R_{ff}(\tau_1 - \tau) e^{-\xi\lambda(t_1-\tau_1)} \sin \lambda_d(t_1 - \tau_1) e^{-\xi\lambda(t-\tau)} \sin \lambda_d(t - \tau) d\tau_1 d\tau, \quad (20) $$

with

$$ \lambda^2 = \omega^2 + 3\alpha\sigma_{ls}^2, \quad (21) $$

$$ \xi = \eta\omega/\lambda, \quad (22) $$

$$ \lambda_d = \lambda(1 - \xi^2)^{1/2}. \quad (23) $$

Since the input is a stationary process equation (20) can be expressed in terms of the input spectral density as

$$ {}_1R_{xx}(t, t_1) = \int_{-\infty}^{\infty} \phi_{ff}(\Omega)|H(\Omega)|^2 [e^{i\Omega(t-t_1)} - e^{-\xi\lambda t}\{(\cos \lambda_d t + \xi_1 \sin \lambda_d t) \cos \Omega t_1 + \xi_1 \sin \lambda_d t \sin \Omega t_1\} $$
$$ - e^{-\xi\lambda t_1}\{(\cos \lambda_d t_1 + \xi_1 \sin \lambda_d t_1) \cos \Omega t + \xi_1 \sin \lambda_d t_1 \sin \Omega t\} $$
$$ + e^{-\xi\lambda(t+t_1)}\{\cos \lambda_d t \cos \lambda_d t_1 + [(\Omega^2 + \xi^2 \lambda)/\lambda_d^2] \sin \lambda_d t \sin \lambda_d t_1 $$
$$ + \xi_1 \sin \lambda_d (t + t_1)\}] d\Omega \quad (24) $$

where

$$ \xi_1 = \xi/(1 - \xi^2)^{1/2}. \quad (25) $$

From this an improved estimate of the response variance, which incidentally is non-stationary, is given as

$$ 2\sigma_x^2(t) = \int_{-\infty}^{\infty} \phi_{ff}(\Omega) |H(\Omega)|^2 [1 + e^{2\xi\lambda t}(1 + 2\xi_1 \sin \lambda_d t \cos \lambda_d t) - 2 e^{-\xi\lambda t}\{(\cos \lambda_d t $$
$$ + \xi_1 \sin \lambda_d t) \cos \Omega t + \xi_1 \sin \lambda_d t \sin \Omega t\} + \{[\Omega^2 - \lambda^2(1 - 2\xi^2)]/\lambda_d^2\} \sin^2 \lambda_d^2 t] d\Omega, \quad (26) $$

where $\phi_{ff}(\Omega)$, the input power spectral density function, is

$$ \phi_{ff}(\Omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R_{ff}(\theta) e^{-i\Omega\theta} d\theta \quad (27) $$

and

$$ |H(\Omega)|^2 = 1/[(\lambda^2 - \Omega^2)^2 + (2\xi\lambda\Omega)^2]. \quad (28) $$

If the iterations were to be stopped at this stage, the approximate steady state variance would be

$$ \sigma_s^2 \doteq \int_{-\infty}^{\infty} \phi_{ff}(\Omega) |H(\Omega)|^2 d\Omega. \quad (29) $$

For example, if the input is a white noise with an autocorrelation

$$ R_{ff}(t, t_1) = I\delta(t - t_1), \quad (30) $$

then

$$ \sigma_s^2 \doteq I/4\xi\lambda^3 = \sigma_{ls}^2/(1 + 3\varepsilon\sigma_{ls}^2), \quad (31) $$

with

$$ \varepsilon = \alpha/\omega^2 \quad (32) $$

which compares with results obtained from other procedures [4]. It is obvious that further iterations can be carried out on a computer only. However, if one is interested only in the steady state variance something better could be done as follows. Suppose the response reaches a stationary steady state; then the output variance gradually tends to a constant value. Thus $\sigma_x^2(t)$ may be taken as a slowly varying function of time and equations (13) and (14) may be treated as constant coefficient equations with $\sigma_x^2(t)$ and $\sigma_x^2(t_1)$ replaced by σ_s^2. When this argument is accepted as plausible it follows that

$$\sigma_s^2 = \int_{-\infty}^{\infty} \phi_{ff}(\Omega)\,|H_e(\Omega)|^2\,\mathrm{d}\Omega, \tag{33}$$

where

$$|H_e(\Omega)|^2 = 1/[(\lambda_e^2 - \Omega^2)^2 + (2\xi_e\lambda_e\Omega)^2], \tag{34}$$

$$\lambda_e^2 = \omega^2 + 3\alpha\sigma_s^2, \tag{35}$$

$$\xi_e\lambda_e = \eta\omega. \tag{36}$$

This equation is more informative than equation (29) since it leads to a polynomial or a transcendental equation with σ_s^2 as the unknown. The roots would indicate the existence or otherwise of many values for σ_s^2 and a possible jump phenomenon. To illustrate this point three different stationary inputs will be considered.

3.1. WHITE NOISE INPUT

In this extreme case, in which the input bandwidth is infinite, equation (33) can be neatly simplified as

$$\sigma_s^2 = I/4\xi_e\lambda_e^3 = \sigma_{ls}^2/(1 + 3\varepsilon\sigma_s^2). \tag{37}$$

The only positive solution of this equation is

$$\sigma_s^2 = \frac{1}{6\varepsilon}\,[(1 + 12\varepsilon\sigma_{ls}^2)^{1/2} - 1]. \tag{38}$$

This is somewhat different from and is hopefully better than equation (31).

3.2. ZERO BANDWIDTH INPUT

At the other extreme is an input which contains only a single frequency. One such excitation is

$$f(t) = a\sin\omega_c t + b\cos\omega_c t, \tag{39}$$

where a and b are independent Gaussian random variables with mean zero and variance σ_f^2. The input autocorrelation is

$$R_{ff}(t, t_1) = \sigma_f^2\cos\omega_c(t - t_1). \tag{40}$$

The corresponding power spectral density is

$$\phi_{ff}(\Omega) = \sigma_f^2[\delta(\omega_c - \Omega) + \delta(\omega_c + \Omega)]. \tag{41}$$

From equation (33) it follows that

$$\sigma_s^2 = \sigma_f^2/[(\lambda_e^2 - \omega_c^2)^2 + (2\xi_e\lambda_e\omega_c)^2]. \tag{42}$$

This easily may be recognized as a cubic equation in σ_s^2. The situation is very similar to the behaviour of the system under a deterministic sinusoidal excitation exhibiting the well known jump characteristics [12]. To study equation (42) it is expedient to treat it as a quadratic in (ω_c/ω) rather than to analyze it as a cubic in σ_s^2. Solving for (ω_c/ω), one gets

$$(\omega_c/\omega) = [(1 + 3\varepsilon\sigma_s^2 - 2\eta^2) \pm \{\sigma_f^2/(\omega^4\sigma_s^2) - 4\eta^2(1 + 3\varepsilon\sigma_s^2 - \eta^2)\}^{1/2}]^{1/2}. \tag{43}$$

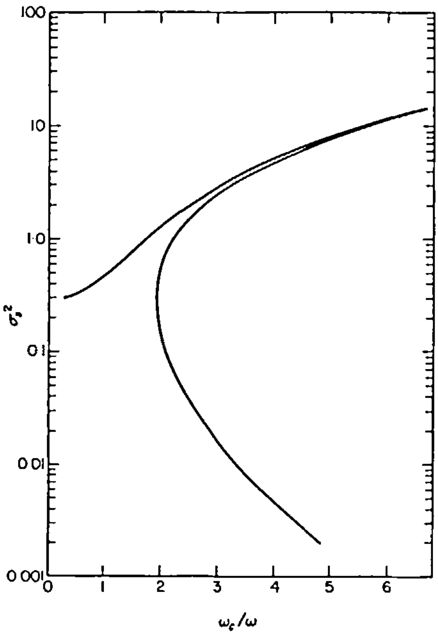

Figure 1. Jump phenomenon of the mean square response for a single frequency random input.

In Figure 1, this has been plotted for $\varepsilon = 1$, $\eta = 0{\cdot}02$, and $\sigma_f^2/\omega^4 = 1$, (ω_c/ω) being considered as a function of σ_ξ^2. This curve illustrates the jump phenomenon of the mean square response spectacularly.

3.3. NARROW BAND INPUT

The existence of the mean square jump characteristics in non-linear systems has been explored previously by Lyon *et al.* [13] to a limited extent. They observed that jumps are possible with narrow band excitations. The results of the previous case also indicate that this may be possible for very narrow bandwidth inputs. Also narrow band inputs are more realistic than the two types discussed previously. With these points in view a band limited white noise will be considered in some detail.

The input spectral density is

$$\phi_{ff}(\Omega) = S_0, \quad \omega_1 < |\Omega| < \omega_2,$$
$$= 0, \quad \text{otherwise}. \tag{44}$$

Hence [14]

$$\sigma_\xi^2 = 2S_0 \int_{\omega_1}^{\omega_2} |H(\Omega)|^2 \, d\Omega = (\pi S_0/2\xi_e \lambda_e^3)\,[I(\omega_2/\lambda_e, \xi_e) - I(\omega_1/\lambda_e, \xi_e)], \tag{45}$$

where

$$I(\mu/\lambda_e, \xi) = \frac{1}{\pi}\tan^{-1}\left[\frac{2\xi_e(\mu/\lambda_e)}{1-(\mu/\lambda_e)^2}\right] + [\xi_e/2\pi(1-\xi_\theta^2)^{1/2}]\log\left[\frac{1+(\mu/\lambda_e)^2+2(1-\xi_e^2)^{1/2}(\mu/\lambda_e)}{1+(\mu/\lambda_e)^2-2(1-\xi^2)^{1/2}(\mu/\lambda_e)}\right]. \tag{46}$$

By taking

$$\omega_1 = (\omega_c - \mathrm{j}\Delta\omega), \quad \omega_2 = (\omega_c + \mathrm{j}\Delta\omega). \tag{47}$$

and

$$\Delta\omega = \eta\omega = \xi_e \lambda_e, \tag{48}$$

equation (45) can be reduced to the form

$$\frac{(\sigma_f^2/\sigma_s^2)}{8j\eta^2\omega^4 z^2}\left[\tan^{-1}\left\{\frac{4j\eta^2(z^2+\delta^2-j^2\eta^2)}{(z^2-j^2\eta^2+\delta^4-2\delta^2(z^2+j^2\eta^2)}\right\}+\frac{\eta}{(z^2-\eta^2)^{1/2}}\log\right.$$

$$\left.\left\{\frac{z^2+(\delta+j\eta)^2+2(z^2-\eta^2)^{1/2}(\delta+j\eta)}{z^2+(\delta+j\eta)^2-2(z^2-\eta^2)^{1/2}(\delta+j\eta)}\,\frac{z^2+(\delta-j\eta)^2-2(z^2-\eta^2)^{1/2}(\delta-j\eta)}{z^2+(\delta-j\eta)^2+2(z^2-\eta^2)^{1/2}(\delta-j\eta)}\right\}\right]=1, \quad (49)$$

where

$$\sigma_f^2=2s_0(\omega_2-\omega_1),\ \delta=\omega_c/\omega,\quad z=(1+3\varepsilon\sigma_s^2). \quad (50)$$

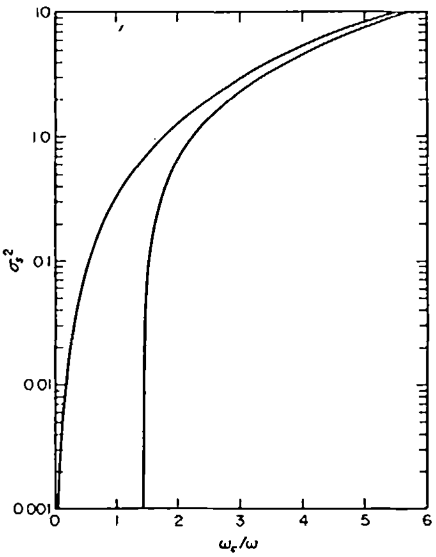

Figure 2. Jump phenomenon of the mean square response for a narrow band input.

This equation has to be solved for various values of σ_s^2 with (ω_c/ω) treated as the unknown, to construct the frequency response curves. In Figure 2, such a result is presented for $\varepsilon=1$, $\sigma_f^2/\omega^4=1$ and $j=1$. It is observed that for any of the values of (ω_c/ω) there could be at the most two values of σ_s^2, unlike the situation in the previous case. Further questions of stability and the effect of the bandwidth are not taken up here. A conjecture based on the known results of the deterministic theory would be that the right branch of Figure 2 is unstable. It may be noted here that the jump phenomenon discussed refers only to the mean square values in the steady state. Associated with the mean square jumps there also would be observable jumps in the response samples. The experiments of Lyon *et al.* [13] qualitatively verify this observation. Detailed investigations are needed to make more conclusive statements.

4. PROBABILITY DISTRIBUTION OF THE RESPONSE

The distribution of the response of the non-linear system under consideration is going to be obviously non-Gaussian. At present no exact solutions are known for this except when the input is a white noise, that too only in the steady state. In what follows, an attempt has been made to arrive at the joint density function of x and $\dot{x}$, for any Gaussian input, in an heuristic manner.

The previous analysis for the response autocorrelation hinged essentially on the assumption that the response be approximately Gaussian, but with unknown parameters which are to be found from equations (13) and (14). Although the non-linearity inherent in the original system is reflected in these equations, nonetheless the autocorrelation one gets refers strictly to a hypothetical Gaussian response process, $\tilde{x}(t)$, and hence to a quasi-linear system with a time varying coefficient $[\omega^2 + 3\alpha\sigma_1^2(t)]$. Incidentally, this observation indicates that there is some resemblance between the present method and the equivalent linearization technique. However, the error minimization essential to the latter has not been invoked at all here. Once the autocorrelation, $R_{\tilde{x}\tilde{x}}(t, t_1)$, of the process, $\tilde{x}(t)$, is known the moments of the derivative process, $\dot{\tilde{x}}$, can be found. This would lead directly to the joint density function, $p(\tilde{x}, \dot{\tilde{x}}; t)$, since this quantity is Gaussian. The hypothetical system leading to $\tilde{x}(t)$ can be described by the equation

$$\ddot{\tilde{x}} + 2\eta\omega\dot{\tilde{x}} + [\omega^2 + 3\alpha\sigma_1^2]\tilde{x} = f(t). \qquad (51)$$

If the original system had had non-linear damping, then the above equation would have had time varying coefficients with $\dot{\tilde{x}}$ also. Since it is this equation that has been solved and not equation (1), it would be natural to ask in what sense $(x, \dot{x})$ and $(\tilde{x}, \dot{\tilde{x}})$ compare. Instead of posing such a question on the response variable, herein, the two systems—the real non-linear and the hypothetical linear—may be brought nearer, in a different sense. It is clear that the nearness of the two systems hints at an equivalence criterion. Since the responses are obviously different one would look for an equivalence of the functions of the response variables rather than expect x and $\tilde{x}$ to compare. One such criterion, which is reasonable, is the equivalence of the energies in the two systems at any time. The energy in the given non-linear system is

$$E_{nl} = m\int_0^x \ddot{x}\,\mathrm{d}x + c\int_0^x \dot{x}\,\mathrm{d}x + \int^x (kx + \alpha m x^3)\,\mathrm{d}x$$

$$= m\frac{\dot{x}^2}{2} + c\int_0^t \dot{x}^2\,\mathrm{d}t + \frac{kx^2}{2} + \alpha m\frac{x^4}{4}. \qquad (52)$$

Similarly the energy in the hypothetical system is

$$E_{hl} = m\frac{\dot{\tilde{x}}^2}{2} + c\int_0^t \dot{\tilde{x}}^2\,\mathrm{d}t + k(1 + 3\varepsilon\sigma_1^2)\frac{\tilde{x}^2}{2}. \qquad (53)$$

Here, m, c and k refer to the mass, dash pot constant and the coefficient of the linear part of the spring, respectively. Inspection of equations (52) and (53) suggests the simple memory-less transformation

$$x^2 + \frac{\varepsilon}{2}x^4 = (1 + 3\varepsilon\sigma_1^2)\tilde{x}^2, \qquad (54)$$

$$\dot{x} = \dot{\tilde{x}}, \qquad (55)$$

to establish the equivalence of the two energies. Now, since $p(\tilde{x}, \dot{\tilde{x}}; t)$ is known, $p(x, \dot{x}; t)$ easily can be determined. In the present problem

$$p(\tilde{x}, \dot{\tilde{x}}; t) = \frac{1}{2\pi\sigma_1\sigma_2(1 - r^2)^{1/2}}\exp\left[-\frac{1}{2(1 - r^2)}\left\{\frac{\tilde{x}^2}{\sigma_1^2} - \frac{2r\tilde{x}\dot{\tilde{x}}}{\sigma_1\sigma_2} + \frac{\dot{\tilde{x}}^2}{\sigma_2^2}\right\}\right], \qquad (56)$$

where,

$$\sigma_1^2 = \langle\tilde{x}^2(t)\rangle, \quad \sigma_2^2 = \langle\dot{\tilde{x}}^2(t)\rangle, \quad r(t) = \langle\tilde{x}(t)\dot{\tilde{x}}(t)\rangle/\sigma_1\sigma_2. \qquad (57)$$

From equation (54),

$$\tilde{x} = \pm(x^2 + \varepsilon x^4/2)^{1/2}/(1 + 3\varepsilon\sigma_1^2)^{1/2} \qquad (58)$$

and

$$|\mathrm{d}\tilde{x}/\mathrm{d}x| = (1+\varepsilon x^3)/[(1+3\varepsilon\sigma_1^2)]^{1/2}(1+\varepsilon x^2/2). \tag{59}$$

Thus,

$$p(x,t) = (1+\varepsilon x^2)/[2\pi\sigma_1^2(1+\varepsilon x^2/2)(1+3\varepsilon\sigma_1^2)]^{1/2}\exp[-(x^2+\varepsilon x^4/2)/(2\sigma_{is}^2)], \tag{60}$$

$$p(\dot{x},t) = (1/2\pi\sigma_2^2)^{1/2}\,\mathrm{e}^{-\dot{x}^2/2\sigma_2^2}, \tag{61}$$

$$p(x,\dot{x};t) = \frac{|\mathrm{d}\tilde{x}/\mathrm{d}x|}{2\pi\sigma_1\sigma_2(1-r^2)^{1/2}}\left[\exp\left\{-\frac{1}{2(1-r^2)}\left[\frac{(x^2+\varepsilon x^4/2)}{\sigma_1^2(1+3\varepsilon\sigma_1^2)} - \frac{2r\dot{x}(x^2+\varepsilon x^4/2)^{1/2}}{\sigma_1\sigma_2(1+3\varepsilon\sigma_1^2)^{1/2}} + \frac{\dot{x}^2}{\sigma_2^2}\right]\right\}\right]. \tag{62}$$

Since no assumption about the input, other than normality, has been made in arriving at this result it is expected to be valid for both stationary and non-stationary excitations in either the transient or steady state regimes of the response. That this optimism is not unfounded may be seen from the case of a white noise input. The predicted joint density function in the steady state is

$$p(x,\dot{x},\infty) = \frac{|\mathrm{d}\tilde{x}/\mathrm{d}x|}{2\pi\sigma_1\sigma_2}\exp\left[-\frac{(x^2+\varepsilon x^4/2)}{2\sigma_1^2(1+3\varepsilon\sigma_1^2)} - \frac{\dot{x}^2}{2\sigma_2^2}\right]. \tag{63}$$

From equations (37) and (55) it follows that in the steady state

$$\sigma_1^2(1+3\varepsilon\sigma_1^2) = \sigma_{is}^2, \tag{64}$$

$$\sigma_2^2 = \omega^2\sigma_{is}^2. \tag{65}$$

With these simplifications, equation (63) reduces to

$$p(x,\dot{x},\infty) = \frac{|\mathrm{d}\tilde{x}/\mathrm{d}x|}{2\pi\sigma_1\sigma_2}\exp\left[-\frac{1}{2\sigma_{is}^2}(\dot{x}^2/\omega^2 + x^2 + \varepsilon x^4/2)\right], \tag{66}$$

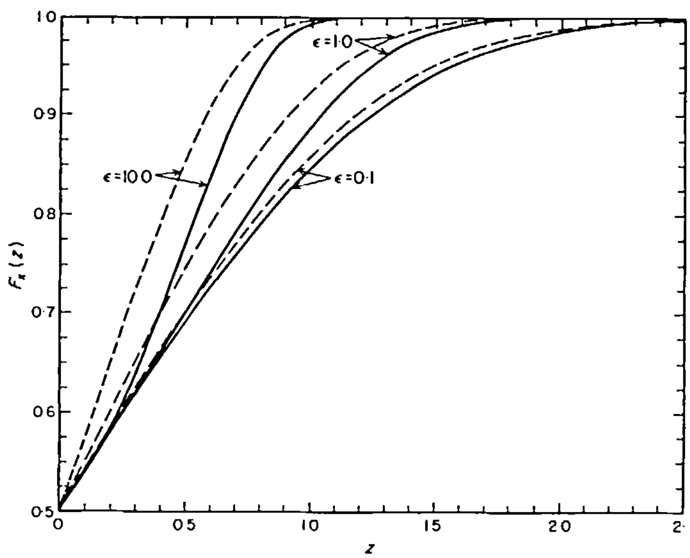

Figure 3. Steady state probability distribution function of x under white noise input. ——, Estimate; -----, exact.

which is very near to the exact solution as obtained from the Fokker–Planck equation [1]. The exact density function of x in the steady state is given by

$$p(x) = C\mathrm{e}^{-(x^2+\varepsilon x^4/2)/2\sigma_{ls}^2}, \tag{67}$$

where C is a normalization constant. In Figure 3 the exact and the approximate steady state distribution functions,

$$F_x(z) = \int_{-\infty}^{z} p(x)\ \mathrm{d}x, \tag{68}$$

are shown for three values of ε. Since the distribution is symmetric about the zero mean value only the right-half has been presented. It is observed that the comparison is good in all the cases. The maximum percentage errors for $\varepsilon = 0{\cdot}1$ and $\varepsilon = 1$ are 1·54 and 5·78, respectively. Even for a non-linearity coefficient as large as $\varepsilon = 10$ the maximum error is only 11·42%.

5. SUMMARY AND CONCLUSION

A method which leads to an approximate probabilistic description of non-linear elastic systems under Gaussian excitations has been presented. The method draws inspiration from the moment closure and the equivalent linearization techniques, but is substantially different from both in the details. The approach may be summarized briefly as follows.

The response probability distribution is assumed to be Gaussian with unknown parameters to start with. The given differential equation is suitably averaged with this assumption to arrive at equations for the autocorrelation function. Invariably a set of coupled equations is obtained where the response variance is locked in as a time varying coefficient. The autocorrelation has to be obtained by an iteration procedure on a digital computer. From this the autocorrelations of the derivatives, if they exist, also can be obtained. Once the correlation matrix is known the joint density function of the response and its derivatives at the same time or at different times can be written since it also is going to be Gaussian. However, these densities refer only to a hypothetical linear system and at best are only approximations to the statistics of the given non-linear system. At this stage an equivalence in terms of the total energy of the hypothetical and the real system is assumed. This gives a memory-less non-linear transformation for the response variables which eventually leads to a non-Gaussian density function which is better than the original Gaussian approximation.

The damped Duffing equation with cubic non-linearity in the spring has been considered in some detail. The jump phenomenon associated with the steady state variance for stationary inputs has been investigated and some numerical results also are presented. A very general expression has been obtained for the joint density of the displacement and velocity. In deriving this no assumptions have been made as to the stationarity or otherwise of either the input or the output. Hence the expression is expected to be widely applicable. Other types of non-linearities in the spring and the case of non-linear damping can be handled on similar lines. But, when the damping is non-linear, although the response correlation may be found and the equivalence relations established, the actual determination of the non-Gaussian joint distribution may be very difficult. However, the approximate one-dimensional density function of the response can be found without much difficulty.

The estimated distribution is an asymptotic result definitely valid for small non-linearities. This follows since equation (62) leads to the exact linear density function as $\varepsilon \to 0$. This behaviour also is observed in Figure 3, where for small ε the estimate compares excellently with the exact results. This figure also indicates that the present estimate errs on the conservative side in estimating the probability of $|x|$ exceeding a safe operating level. Thus, it is believed that the estimate would be of great help in further studies on level crossing and peak distribution.

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