Indian Journal of History of Science (2021) 56:71–84
https://doi.org/10.1007/s43539-021-00017-x

ORIGINAL PAPER

Elegant dissection proofs for algebraic identities in Nīlakaṇṭha's *Āryabhaṭīyabhāṣya*

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Received: 5 March 2021 / Accepted: 21 June 2021 / Published online: 12 November 2021

Abstract

Renowned Indian astronomer and mathematician Nīlakaṇṭha Somayājī is well known for his innate ability to provide ingenious proofs. In his elaborate commentary on *Āryabhaṭīya* called *Āryabhaṭīyabhāṣya*, we find elegant *upapattis* or rationales for three algebraic identities involved in calculating cubes and cube-roots. In this paper, we detail these *upapattis* which may be called dissection proofs in the modern parlance. Incidentally, Nīlakaṇṭha's simple, yet concise and convincing demonstrations are pertinent in the context of mathematics pedagogy as well.

Keywords *Āryabhaṭīyabhāṣya* · Nīlakaṇṭha Somayājī · Dissection proof · *Upapatti* · *Yukti* · Cube · Cube-root · Mathematics education

1 Introduction

Being guided by the principle of parsimony (*lāghava*) that has been particularly emphasised in the grammatical tradition,[1] Indian mathematicians and astronomers have long adopted a style of composition in which they succinctly lay down only the rules and procedures in the main text, often at the expense of laying out the rationales and examples. This of course, should not leave an impression in the minds of the readers that the authors either did not know the rationales, or were not compelled to delve deep into them. In Indian tradition, it seems to have been incumbent upon the commentators to discuss the details of the mathematical rules presented in the source texts at length, propound and demonstrate them with examples (*udāharaṇa*) and so on.

Non-cognizance of this aspect, owing to lack of familiarity or otherwise, has led many scholarly works in history of mathematics to opine that Indian mathematics is bereft of any notion of proof (Kline, 1973, p. 190) or to make assertions that Indian mathematicians did not have any sense of logical rigour (Boyer, 1959, pp. 61–62). In recent scholarly works, Srinivas (2005) and Ramasubramanian (2011) have contested these notions and brought to light how several commentaries written on major texts of Indian mathematics and astronomy present rationales, generally called *upapattis* or *vāsanās* for the results and procedures enunciated in the text.

To add to the inventory of *upapattis* discussed in the above papers and elsewhere, here we present a few *upapattis* given by Nīlakaṇṭha in connection with the procedure for finding the cube or cube-root of a given number which is based upon a certain algebraic identity. The organization of the paper is as follows: We first present the etymology of the word *upapatti* in Sect. 2 and then move on to provide a brief survey of *upapattis* in Indian mathematical texts in Sect. 3. A short introduction to Nīlakaṇṭha Somayājī and his *Āryabhaṭīyabhāṣya* is presented in Sect. 4. Following that, as a precursor to discussing Nīlakaṇṭha's proof we briefly discuss the relevant verse in the source text along with the descriptive commentary presented by Nīlakaṇṭha. Then, we provide the demonstration of the proof as enunciated by Nīlakaṇṭha in Sect. 6. Therein, we understand how this dissection proof and the underlying understanding is reflected in the algorithm for deriving the cube root of a number, as presented by Āryabhaṭa. Then, we also discuss dissection proof of another algebraic identity described by Nīlakaṇṭha in Sect. 7. Section 8 ends with a few concluding remarks.

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[1] As the adage goes:
अर्धमात्रालाघवेन पुत्रोत्सवं मन्यन्ते वैयाकरणाः।

2 The meaning of the word *Upapatti*

The notion of *upapatti* is significantly different from the notion of 'proof' which is understood as a formal axiomatic deductive system. The word *upapatti* can etymologically be derived from the verbal root (*dhātu*) '*pad*' which means 'to go' or 'to attain'. By adding the prefix '*upa*' and the suffix '*ktin*' we get the desired form:

उप + पद् + क्तिन् = उपपत्तिः ।

The prefix '*upa*' is used to convey proximity or closeness. As per the *sūtras* in the Paṇinian grammar, the suffix '*ktin*' is supposed to be employed in '*bhāvārtha*' (having the sense of the verbal meaning). Thus the word *upapatti* literally means "attaining close to". Additionally this suffix can also be taken in *karaṇārtha* as explained in *Mahābhāṣya* and *Vārtika*.[2] This *vārtika* essentially states that the relaxation that is given for the *kṛtya*– suffixes (*kṛtya-pratyayas*)—to be used in the sense of other *kārakas* than the *bhāvārtha*, by the use of the word *bahula*—can be extended to *kṛt*-suffixes also. Since '*ktin*' belongs to this class of suffixes (*kṛt-pratyayas*), we have the license to use it in *karaṇārtha*, which gives a lot of sense to the word *upapatti*. Thus the word *upapatti* can be taken to convey the meaning "that which takes you much closer to understanding [of the subject matter under discussion]." Here the phrase 'moving closer' [to knowledge] is a metaphor to convey 'ascertaining validity' of the knowledge that has been gained. In other words, *upapattis* or *yuktis* enable us to convince ourselves about the verity of a given statement.

In the Indian philosophical tradition, *upapattis* form a set of coherent logical arguments that justify a hypothesis or any statement that needs to be substantiated in a context. The definition of the term *upapatti* provided by the 15th century philosopher Sadānanda may be worth recalling here. Towards the end of his short, yet popular text on *Advaita Vedānta* called *Vedāntasāra* he notes:

> प्रकरणप्रतिपाद्यार्थसाधने तत्र तत्र श्रूयमाणा युक्तिः उपपत्तिः ।
>
> *prakaraṇapratipādyārthasādhane tatra tatra śrūyamāṇā yuktiḥ upapattiḥ* ।
>
> *upapatti* is [essentially] the reasoning that is adduced at different places in support of something that needs to be elucidated or convinced in a given context (*prakaraṇa*).

The use of the word *prakaraṇa* in the above definition is worth noting. It clearly points to the fact that *upapatti* cannot be conceived to be an entity that is universal, but can only be contextual. In fact, it not only depends upon the context, but also varies with time and subject-matter or the discipline under discussion.

In Indian Mathematics, *upapattis* would entail one or more of the following: *upapatti* in the form of logical sequence of arguments, *upapatti* in the form of geometric demonstration and *upapatti* in the form of mathematical analysis. An illustration for each of this type has been presented by Ramasubramanian (2011).

3 A brief survey of the texts presenting *Upapattis*

Srinivas (2005) has presented a list of texts that involve proofs in an appendix, commenting on the tradition of providing mathematical *upapattis* in India. The earliest exposition of *upapattis* in Indian mathematical and astronomical works dates back at least to the time of Govindasvāmin (c. 800 CE) and and Caturveda Pṛthūdakasvāmin (c. 860 CE). In the works of Bhāskarācārya (b. 1114 CE) very skillful expositions of *upapattis* are found. In the medieval period, the commentaries of Nīlakaṇṭha Somayājī (b. 1444 CE), Śāṅkara Vāriyar (c. 1535 CE), Gaṇeśadaivajña (c. 1545 CE), Kṛṣṇadaivajña (c. 1600 CE) and the famous Malayalam work *Yuktibhāṣā* of Jyeṣṭhadeva (1530 CE) contain many instances of detailed *upapattis*.

Some of these *upapattis* were noted in the early European studies on Indian mathematics in the first half of the nineteenth century. For instance, in 1817, H.T. Colebrooke (1837, p. 439) notes the following in the preface to his widely circulated translation of portions of *Brāhmasphuṭasiddhānta* of Brahmagupta and *Līlāvatī* and *Bījagaṇita* of Bhāskarācārya:

> On the subject of demonstrations, it is to be remarked that the Hindu mathematicians proved propositions both algebraically and geometrically: as is particularly noticed by Bhāskara himself, towards the close of his algebra, where he gives both modes of proof of a remarkable method for the solution of indeterminate problems, which involve a factum of two unknown quantities.

Among this galaxy of commentators who also have produced phenomenal original works, Nīlakaṇṭha Somayājī in his commentary *Āryabhaṭīyabhāṣya* has provided an elaborate *upapattis* that are both engaging and sophisticated. Besides presenting *upapattis* for various mathematical formulae, Nīlakaṇṭha has also tactfully presented incisive logical arguments to deduce the heliocentric motion of Mercury and Venus. In the mathematical context, he seems to have a proclivity to present elegant geometric proofs for summation relations as shown by Mallaya (2001), Mallayya (2002) and Ramasubramanian (2011). In the context of employing

[2] In the commentary of the *sūtra* कृत्यल्युटो बहुलम् (3.3.113) we find the *vārtika* कृतो बहुलम् पादहारकाद्यर्थम्।

geometrical *upapattis*, Saraswati Amma (1999, p. 23) extols Nīlakaṇṭha and some of his contemporaries by stating that:

> The full bloom of this geometrical-algebraical imagination [is] found in Nīlakaṇṭha Somayājī and his followers, the authors of the *Kriyākramakarī* and the *Yuktibhāṣā*.

In what follows, we present the ingenious dissection proof for three specific algebraic identities relating to computing cubes and cube-roots that we find in the *Gaṇitapāda* of his *Āryabhaṭīya-bhāṣya*.

4 About *Āryabhaṭīya-bhāṣya* and its author

It is widely known that Āryabhaṭa was an eminent astronomer and mathematician who flourished in the latter half of the 5th century CE. His magnum opus *Āryabhaṭīya* is one among the highly revered works on astronomy and mathematics in India, and which has also inspired several other later works. That it has received wide accolades throughout India can be easily guaged from the fact that an accomplished astronomer of Nīlakaṇṭha's nature sets forth to author a commentary to this work almost a thousand years later after its completion.

Interestingly, this work comprises just 121 verses all in the *Āryā* meter.[3] However, Āryabhaṭa has been successful in encompassing in them a wide range of mathematical topics, parameters for computations and various astronomical computations including planetary positions and eclipses. We find several commentaries on it composed by later astronomers which speaks of both the need and reputation enjoyed by this work. In fact, we are deeply indebted to the commentators of this work but for whose efforts in elucidating the terse and packed verses of Āryabhaṭa, it would be almost impossible for us to appreciate the profundity of *Āryabhaṭīya*. Among the commentaries on the *Āryabhaṭīya* that we have been able to trace so far, the *Āryabhaṭīyabhāṣya* of Nīlakaṇṭha Somayājī is by far the best and most elaborate one.[4]

Nīlakaṇṭha (1444–1544 CE), hailed from Trikkaṇṭiyūr (Kuṇḍagrāma) near Tirūr in south Malabar, a famous seat of learning in Kerala during the middle ages. He is one of the renowned mathematicians and astronomers of the Kerala school of astronomy and mathematics. He was a disciple of Dāmodara, who was the son and disciple of Parameśvara. In his own words, Nīlakaṇṭha refers to Parameśvara as his *Paramaguru* and that he is indebted to him for many results and insights (Ramasubramanian & Sriram 2011, p. 35). We gather from his works that Nīlakaṇṭha was well versed not only in *Jyotiṣa*, but also in other branches of knowledge such as *Mīmāṃsā*, *Nyāya*, *Vedānta* and so on. His known works include *Āryabhaṭīyabhāṣya*, *Golasāra, Tantrasaṅgraha, Siddhāntadarpaṇa, Jyotirmīmāṃsā*, etc.

Nīlakaṇṭha states in his auto-commentary on *Siddhāntadarpaṇa* that he was born on *Kali* day 1660181 which corresponds to June 17, 1444 CE (Mahesh 2010, p. 108; *Siddhāntadarpaṇa* of Nīlakaṇṭha Somayājī with autocommentary 1977). That he lived to a ripe old age, even to become a centenarian, is attested by a reference to him in *Praśnasāra*, a Malayalam work on astrology. The erudition of Nīlakaṇṭha in several branches of Indian philosophy including other scriptures such as *Dharmaśāstras*, *Purāṇas*, and so on, is quite evident from the frequent references to them in his works, particularly *Āryabhaṭīyabhāṣya* and *Jyotirmīmāṃsā*. This is in addition to the citations from *Jyotiṣa* works beginning from *Vedāṅga-jyotiṣa* down to the treatises of his own times.

The *Āryabhaṭīyabhāṣya* composed by Nīlakaṇṭha late (*pravayasā*) in his life[5] is yet to be fully translated and studied in detail. He himself calls it a *Mahābhāṣya*, which is amply justified considering the wealth of information and very detailed explanations available in it. In a sense, this work mirrors the prevalent knowledge of mathematics and astronomy in India in general, and Kerala in particular. He also supplements it with his own insights. This work also incorporates various leaps made in astronomy including the geometrical model of planetary motion, eclipses and even *upapattis* including deduction of the heliocentric motion of Mercury and Venus (Ramasubramanian et al. 1994).

Nīlakaṇṭha presents multi-fold reasoning to the enunciations of Āryabhaṭa along with a number of citations of authority, illustrations and various related topics. Presenting more details and insights into those matters that are only briefly touched upon in the original text and providing detailed rationales of different rules are among the features that are entailed upon the commentary. One such instance found in *Āryabhaṭīyabhāṣya* in connection with the mathematical procedure of cube-root extraction is what we are presenting in this paper. Before proceeding to the

[3] Some of the verses are in *Gīti* meter which essentially comes under the *Āryā* class of moric meters.

[4] This is not to undermine the significance of the commentary of Bhāskara I (7th cent.), which is also an extremely important and elaborate commentary. What may be worth noting is the fact that the nature, style and emphasis of the two commentaries widely vary from one another.

[5] The following statement of Nīlakaṇṭha appears in his commentary on verse 26 of the *Gaṇita* section (*Āryabhaṭīya of Āryabhaṭācārya* 1930, p. 156):

> ... मयाद्य प्रवयसा ज्ञाता युक्तीः प्रतिपादयितुं भास्करादिभिः अन्यथा व्याख्यातानां कर्माण्यपि प्रतिपादयितुं यथाकथञ्चिदेव व्याख्यानमारब्धम्।

> ... somehow, I have started the commentary today at my ripe age, in order to present the rationales that have been understood by me, and also to describe the procedures explained differently by Bhāskara, etc.

proof we would like to briefly touch upon the emphasis that Nīlakaṇṭha lays on presenting *upapattis*.

5 Nīlakaṇṭha's insistence on providing rationales and demonstrations

Primarily influenced by the study of some of the works of Nīlakaṇṭha, the renowned scholar and scientist Roddam Narasimha, in his highly erudite article (Narasimha 2012) argues how Indian astronomers and mathematicians greatly valued *yuktis* in order to acquire what may be called as "reliable knowledge". There, Narasimha anchors his argument by citing Nīlakaṇṭha's other works such as *Siddhāntadarpaṇa*. Here, in this section, by quoting from the *Āryabhaṭīya-bhāṣya* we show how Nīlakaṇṭha has placed immense importance to meticulously present rationales, generally referred to as *yuktis* or *upapattis* of different rules and procedures that we find in mathematics or astronomy. A clear testimonial to this style of Nīlakaṇṭha is evident from the following statement in his commentary (Śāstrī 1930, p. 28).

राश्योर्योगः तदन्तरेण गुणितः तयोर्वर्गान्तरं स्यादिति।
युक्तिश्चोभयथा प्रदर्श्या – गणनन्यायमात्रेण क्षेत्रकल्पनया च। तत्र छेद्यके वैशद्यं स्यात्।

rāśyoryogaḥ tadantareṇa guṇitaḥ tayorvargāntaraṃ syāditi |
yuktiścobhayathā pradarśyā – gaṇananyāyamātreṇa kṣetrakalpanayā ca| tatra chedyake vaiśadyaṃ syāt |

The sum of two numbers multiplied by their difference would be [equal to] the difference of their [individual] squares. The rationale should be demonstrated both ways—by the rules of arithmetic and algebra (*gaṇana*) as well as by geometric constructions. In the geometric construction [method] (*chedyaka*) there will be clarity (*vaiśadya*).

The use of word *pradarśyā* is noteworthy here. In order to better appreciate why it has been employed by Nīlakaṇṭha, it may be worthwhile to see its grammatical derivation:

प्र + दृश् + ण्यत् → प्र + दृश् + य
→ प्रदर्श्य
प्रदर्श्य + टाप् → प्रदर्श्या

Here, the suffix '*ṇyat*' that has been added to the verbal root *dṛś* (to see), belongs to a class of suffixes known as *kṛtya-pratyayas*. They have the potential to convey that something is "ought to be done" (*praiṣārtha*).[6] Thus, one can see that Nīlakaṇṭha strongly emphasises that the rationale behind various mathematical rules must/ought to be demonstrated by the teachers. Furthermore, his statement to use mathematical reasoning (*gaṇana-nyāya*) as well as geometric constructions (*kṣetra-kalpanā*) mirrors his intent in creating knowledge base that is reliable, elegant and accessible to learners of all age groups whose abilities to grasp things widely vary.

In another instance, Nīlakaṇṭha shows his proclivity to go further and present demonstrations using building blocks made out of clay in order to make things far simpler for children to appreciate the rationale.

तुल्यानां विस्तृतिदीर्घपिण्डाना घातो घनः। तद्युक्तिरपि मृदादिना प्रदर्श्या ॥

tulyānāṃ vistṛtidīrghapiṇḍānāṃ ghāto ghanaḥ |
tadyuktirapi mṛdādinā pradarśyā ||

The cube is the product of breadth, length, and width that are equal. Also, its rationale has to be demonstrated by making use of a lump of clay, etc.

The use of the word *mṛdādi* gives us a cue to the fact that the demonstrations of the rules were provided not just through clay models but other means too. Nonetheless, it is certain that Nīlakaṇṭha has had a strong disposition to provide elegant geometric proofs, wherever it was possible to do so. As stated earlier, the objective of this paper is to bring out the elaborate geometrical construction, which may also be described as dissection proof, provided by Nīlakaṇṭha to substantiate the validity of an algebraic identity connected with the mathematical process of cubing a number or inversely the process of extracting the cube-root from it.

With this backdrop, we shall now delve into the details of the *upapattis* offered by Nīlakaṇṭha.

6 Proof demonstrated by Nīlakaṇṭha

6.1 Definition of cube

Since this paper deals with Nīlakaṇṭha's commentary on *Āryabhaṭīya*, it would only be appropriate to commence our discussion with the verse of Āryabhaṭa that defines what a cube is. Āryabhaṭa who is ingenious and matchless in his ability to densely pack enormous amount of information in a single verse provides the following definition of a cube right at the beginning of the chapter on *Gaṇita* (Shukla 1976, p. 35):

सदृशत्रयसंवर्गो घनः तथा द्वादशाश्रिः स्यात्॥3॥

sadṛśatrayasaṃvargo ghanaḥ tathā dvādaśāśriḥ syāt ||3||

The product of three equals as also the solid having twelve edges is a cube.

[6] This is as per the *sūtra* of Pāṇini: प्रैषातिसर्गप्राप्तकालेषु कृत्याश्च (3.3.163).

It is noteworthy that this short definition (*āryārdha*) encompasses both the arithmetic operation involved in finding the cube of a number as well as its geometric equivalent. Following this verse, we find the procedure for finding the cube-root of a given number described in a single verse (Shukla 1976, p. 37):

अघनाद् भजेद् द्वितीयात् त्रिगुणेन घनस्य मूलवर्गेण ।
वर्गस्त्रिपूर्वगुणितः शोध्यः प्रथमाद् घनश्च घनात् ॥5॥

aghanād bhajed dvitīyāt triguṇena ghanasya mūlavargeṇa |
vargastripūrvaguṇitaḥ śodhyaḥ prathamād ghanaśca ghanāt ‖5‖

(Having subtracted the greatest possible cube from the last cube place and then having written down the cube-root of the number subtracted in the line of cube-root), divide the second non cube place by thrice the square of the root (of the subtracted cube). Then subtract the square of the quotient multiplied by thrice the previous [root] and, cube from the cube place.

For a detailed explanation of the procedure given in the verse above the reader is referred to the scholarly edition of the text brought out by K. S. Shukla (1976, p. 37) whose translation is furnished above. It would suffice to mention here that the rationale behind the procedure for finding either cube or cube-root of a given number crucially depends upon the following algebraic identity.

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3. \quad (1)$$

6.2 Nīlakaṇṭha's *upapatti* for the algebraic identity

It is important to note that Nīlakaṇṭha, truly playing the role of an expert commentator, first introduces the identity (1), since Āryabhaṭa does not mention it, and then presents how the expansion of the identity can be obtained through *khaṇḍa-guṇana* (multiplication by parts). Having detailed the formulation of the identity he also connects us to a verse from *Līlāvatī*. In order to have a better appreciation of Nīlakaṇṭha's commentary, the following would help in getting introduced to a few terminologies employed by him.

Let N be the number whose cube is to be determined. Let it be written as the sum of two other numbers say a and b. That is,

$$N = a + b \qquad (\text{with } a < b).$$

The terminology employed by Nīlakaṇṭha to refer to a and b are *alpakhaṇḍa* and *mahākhaṇḍa* respectively. Specifying the terms in the RHS of (1) Nīlakaṇṭha notes:

तस्मादल्पवर्गे त्रिभिर्हते महता च हते अष्टसु त्रयः खण्डाः परिगृहीताः स्युः। महतो वर्गेऽपि त्रिभिरल्पेन च हते त्रयः। खण्डघनाभ्यामपि द्वौ। एवमष्टानां खण्डानां परिग्रहेण घनः कृत्स्न एव सम्पद्यते।

tasmādalpavarge tribhirhate mahatā ca hate aṣṭasu trayaḥ khaṇḍāḥ parigṛhītāḥ syuḥ | *mahato varge'pi tribhiralpena ca hate trayaḥ* | *khaṇḍaghanābhyāmapi dvau* | *evamaṣṭānāṃ khaṇḍānāṃ parigraheṇa ghanaḥ kṛtsna eva sampadyate* |

Therefore when the square of the smaller portion multiplied by three is further multiplied by the larger portion, three out of eight factors [of the final expression] would have been taken care of. Also, the square of the larger portion when multiplied by three and smaller portion will also yield three [factors]. Two [factors] are obtained by the cubes of both portions. Thus by taking care of all the eight parts the [value of the] entire cube (*kṛtsnaḥ khaṇḍaḥ*) is obtained.

In the above passage, Nīlakaṇṭha essentially spells out the factors in the RHS of the identity given by (1).

$$\begin{aligned} N^3 &= (a+b)^3 \\ &= (a+b)\times(a+b)^2 \\ &= (a+b)\times(a^2+ab+ab+b^2) \\ &= a^3 + a^2b + a^2b + a^2b + ab^2 + ab^2 + ab^2 + b^3 \quad (2) \\ &= a^3 + 3a^2b + 3ab^2 + b^3. \quad (3) \end{aligned}$$

6.3 Demonstration of the proof

Though generally commentaries are written in prose, here Nīlakaṇṭha interestingly employs both *gadya* (prose) and *padya* (poetry) while providing this dissection proof. The proof essentially consists of the following steps:

1. Considering a cube of suitable material (such as clay) and dimension that can be easily dissected.
2. Making a few marks on it with appropriate dimensions along a few edges and dissecting the cube along three perpendicular axes.
3. Computing the volume of the eight resulting pieces to demonstrate that they actually correspond to the eight terms in the RHS of Eq. (2).
4. Grouping the identical pieces to show that the four groups that get formed correspond to the four terms in the RHS of (3).

समद्वादशाश्रस्य कस्यचित् घनक्षेत्रस्य अश्राणां तुल्यतया त्रेधा खण्डनं कृत्वा अष्टौ खण्डाः पृथक्कृत्य प्रदर्श्योः। तच्च उदाहरणपुरःसरं प्रदर्शयिष्यामः। तत्र नवविस्तृतिदीर्घपिण्डे द्वादशाश्रे तावत् प्रदर्श्यते।

samadvādaśāśrasya kasyacit ghanakṣetrasya aśrāṇāṃ tulyatayā tredhā khaṇḍanaṃ kṛtvā aṣṭau khaṇḍāḥ *pṛthakkṛtya pradarśyāḥ* | *tacca udāharaṇapuraḥsaraṃ pradarśayiṣyāmaḥ* | *tatra navavistṛtidīrghapiṇḍe dvādaśāśre tāvat pradarśyate* |

By dissecting a cube (*ghanakṣetra*) of twelve equal edges (*samadvādaśāśra*) through three sectional cuts by using the same proportion (i.e., a:b) along all [the three] edges [chosen from any corner], the resulting eight pieces have to be shown by dismantling them. We shall demonstrate it with an example. This is being demonstrated in a cube having breadth, length and height equal to nine [units].

Having described the process of dissecting the cube Nīlakaṇṭha proceeds to graphically describe the nature of the resulting solids as follows:

तत्र नवसङ्ख्यस्य बाहोः चतुस्सङ्ख्य एकः खण्डः, इतरः पञ्चसङ्ख्यः। तत्र भूस्पृष्टादेककोणात् प्रभृति त्रिष्वप्यश्रेषु हस्तचतुष्कमितेऽङ्कं कृत्वा विभक्ते सत्यष्टौ खण्डाः स्युः।

tatra navasaṅkhyasya bāhoḥ catussaṅkhya ekaḥ khaṇḍaḥ, itaraḥ pañcasaṅkhyaḥ | *tatra bhūspṛṣṭādekakoṇāt prabhṛti triṣvapyaśreṣu hastacatuṣkamite'ṅkaṃ kṛtvā vibhakte satyaṣṭau khaṇḍāḥ syuḥ* |

The side of nine units has two parts; one is four, and the other is five (*hastas*) in length. By marking at a distance of four *hastas* on the three edges from one of the corners touching the ground, and by cutting [along the marked lines], eight parts would be obtained.

Having succinctly described in prose, the way a cube has to be dissected, and the nature of the resulting eight pieces, he resorts to explain in great detail how this dissection helps in understanding the rationale behind the algebraic identity (1) by resorting to verses.

It is well known that in metrical form things can be easily committed to memory. With this in mind, Nīlakaṇṭha explains the entire *upapatti* in versified form to facilitate learners to commit the whole of the *upapatti* playfully to memory. Since there are sub-themes within the exposition, in what follows we present the verses under various subsections.

6.3.1 Procedure for dissecting the cube

समद्वादशबाहौ तु विभक्ते च घने[7] त्रिधा ।
युक्तिर्बोध्या विभागाय पृष्ठे रेखाद्वयं लिखेत् ॥1॥
पूर्वापरायतं ह्येकम् अन्यद्याम्योत्तरायतम् ।

samadvādaśabāhau tu vibhakte ca ghane tridhā |
yuktirbodhyā vibhāgāya pṛṣṭhe rekhādvayaṃ likhet ||1||
pūrvāparāyataṃ hyekam anyadyāmyottarāyatam |

[7] Here the word *ghane* is to be understood as *ghanākhye gaṇitakarmaṇi* (in the mathematical procedure for determining cubes).

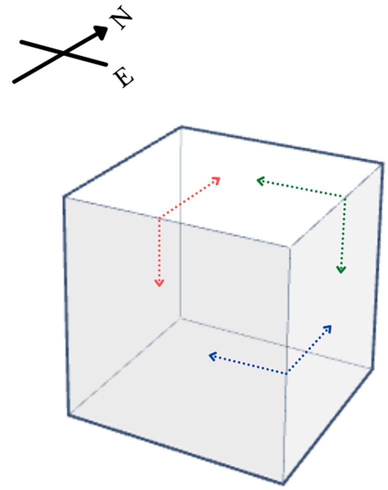

Fig. 1 Three-way dissection of the cube

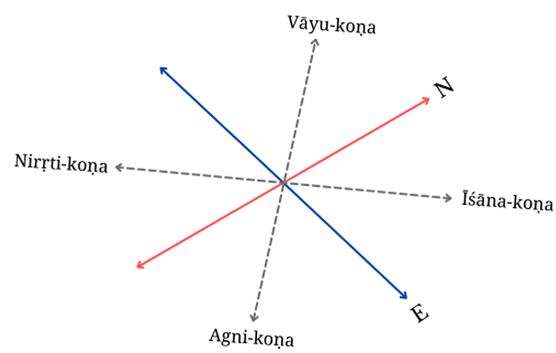

Fig. 2 The inter-cardinal directions and their names

The rationale behind the procedure for finding cubes (*ghane*) can understood by doing a three-fold dissection of a solid (*ghana*) with twelve equal edges. For dissection (*vibhāgāya*), two lines may be drawn on the [top] surface, one extending from east to west and the other from south to north.

अल्पखण्डान्तरे सौम्यात् याम्याच्च महदन्तरे ॥2॥
तथैव प्रत्यगश्राच्च प्रागश्राच्च यथाक्रमम् ।
अल्पखण्डोच्छ्रिते रेखाः कुर्यात् पार्श्वचतुष्टये ॥3॥
विदारिते च तैर्मार्गैरष्टौ खण्डा भवन्ति हि ।

alpakhaṇḍāntare saumyāt yāmyācca mahadantare ||2||
tathaiva pratyagaśrācca prāgaśrācca yathākramam |
alpakhaṇḍocchrite rekhāḥ kuryāt pārśvacatuṣṭaye ||3||
vidārite ca tairmārgairaṣṭau khaṇḍā bhavanti hi |

The lines are drawn such that the east-west line falls at a distance equal to the smaller portion measured from the northern edge and at a distance of the larger portion from the southern edge, and similarly, [the north-south line falls at a distance equal to the smaller portion meas-

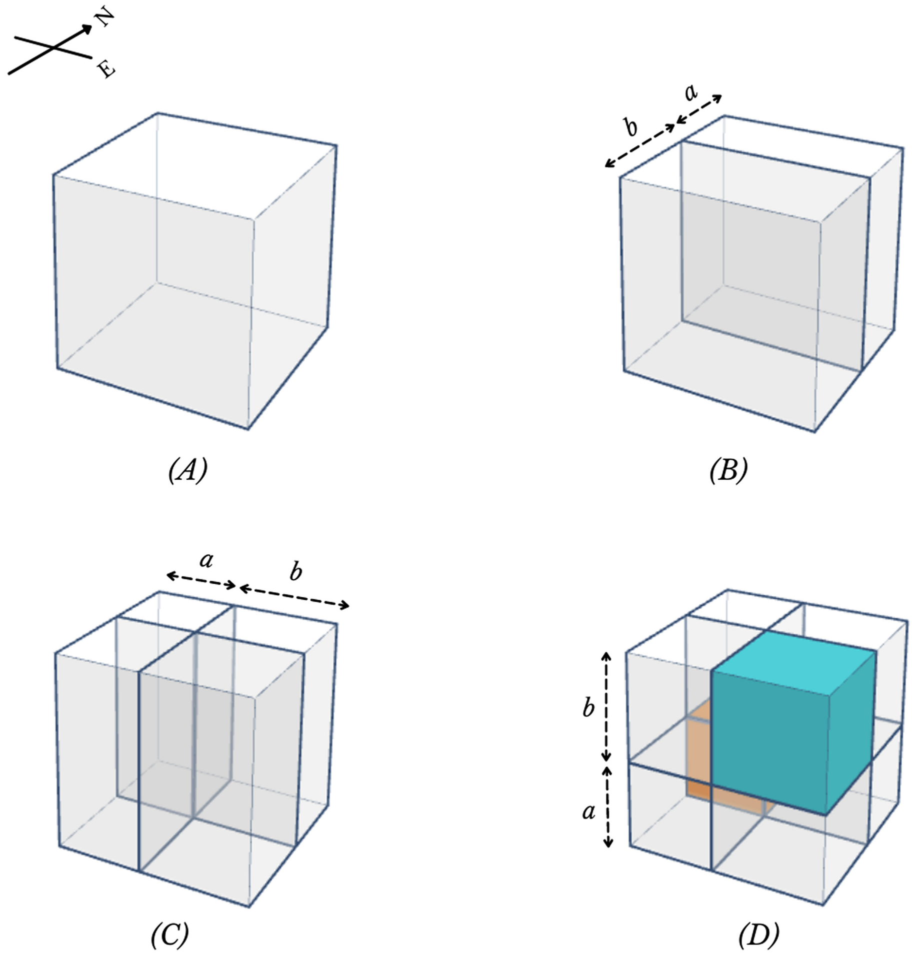

Fig. 3 Depiction of slicing of the cube. **A**. The whole cube, **B**. the first cut along the E-W line, **C**. the second cut along N-S line, **D** the third cross-sectional cut

ured] from the western edge and at a distance of the larger portion from the eastern edge, [and yet another horizontal cross-sectional line] on the upright sides such that it is at a height equal to the smaller portion measured from the ground respectively. When the cube is cut along these lines, there will indeed be eight parts.

The setup involves taking a cube and slicing it as per the instructive directions of Nīlakaṇṭha. Since the algebraic identity (1) for which the proof is being demonstrated involves the sum of two numbers a and b (with $a < b$), the rationale behind dissecting the cube in three ways as prescribed above is quite evident. Two cuts have been made along the cardinal directions and one cut cross-sectionally as indicated in Fig. 1.

Before proceeding further with the explanations of the verses in the later sections, we introduce some of the technical names employed to refer to for inter-cardinal directions. In Fig. 2, we can see that the NW is called *vāyukoṇa*, and SE called *agnikoṇa* and so on. These names have to do with the deities associated with those directions. Since Nīlakaṇṭha uses these names to refer to these directions, we have elaborated on them.

For the purpose of convenience in referring to the resulting eight blocks, we label them as: north-east-top (NET), north-west-top (NWT), south-east-top (SET), south-west-top (SWT), north-east-bottom (NEB), north-west-bottom (NWB), south-east-bottom (SEB), and south-west-bottom (SWB) (see Figs. 3 and 4). Furthermore, to aid our description, we may take the dimension of the cube to be split by the three cuts such that each cut splits the original measure into a (the smaller portion) and b (the bigger portion) as shown in Fig. 3. In the verses that follow, Nīlakaṇṭha explains the dimensions of the cubes and cuboids that have been formed as a result of this dissection.

6.3.2 Smaller cube and its adjacent blocks

अल्पखण्डघनो वायौ[8] भूगतो द्वादशाश्रकः ||4||

alpakhaṇḍaghano vāyau bhūgato dvādaśāśrakaḥ ||4||

[8] वायुकोणे इत्यर्थः।

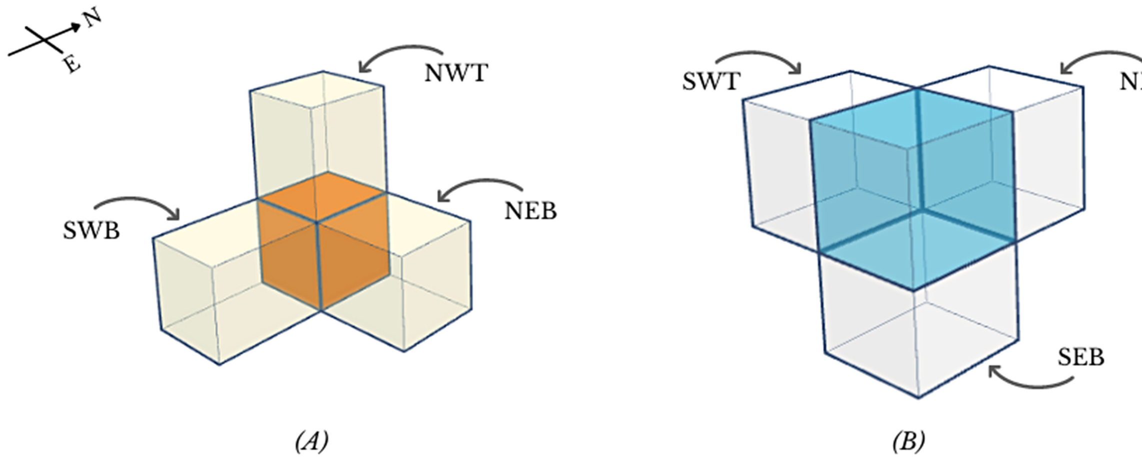

Fig. 4 Blocks adjacent to smaller and bigger cubes. **A**. NEB, SWB and NWT are adjacent to the smaller cube NWB. **B**. NET, SWT and SEB are adjacent to the bigger cube SET

The block whose dimension corresponds to the smaller segment in the northwest direction (*vāyu*) on the ground has twelve [equal] edges [i.e., it is a perfect cube].

ततः प्राग्याम्ययोः खण्डावूर्ध्वगश्च समास्त्रयः ।
अल्पखण्डोच्छ्रिती द्वौ तु महाखण्डोच्छ्रितिः परः ॥5॥

tataḥ prāgyāmyayoḥ khaṇḍāvūrdhvagaśca samāstrayaḥ ।
alpakhaṇḍocchritī dvau tu mahākhaṇḍocchritiḥ paraḥ ॥5॥

The three [blocks] - two towards its east and south, and the one above it are equal. [However], the height of two of them is equal to the short segment (a), and the other to the large segment (b).

Owing to the choice of the lines drawn and cuts made, the NWB block would be a cube with all sides equal to a and its volume would evidently be the smallest among all the other blocks. The two blocks that are adjacent to this cube would be NEB and SWB and their height will be same as that of NWB, which is a. And the block NWT which is right above NWB and its height equal to b. Here, it can be noted that Nīlakaṇṭha guides the learner to observe the smallest block first, and then describes the nature of three other blocks, that are adjacent to it as they share one face with the smallest block is common. This is depicted in Fig. 4a. The term *samaḥ* which literally means equal, has to be understood carefully here. What Nīlakaṇṭha essentially means here is the fact that the volume of the three blocks are equal which is a^2b, as will be explained later.

6.3.3 Bigger cube and its adjacent blocks

ऊर्ध्वभागेऽग्निकोणे यः खण्डः स महतो घनः ।
तदधोगत एकः स्याद् उदक्पार्श्वगतः परः ॥6॥
प्रत्यक्पार्श्वगतोऽन्यश्च त्रय एते मिथः समाः ।

ūrdhvabhāge'gnikoṇe yaḥ khaṇḍaḥ sa mahato ghanaḥ ।
tadadhogata ekaḥ syād udakpārśvagataḥ paraḥ ॥6॥
pratyakpārśvagato'nyaśca traya ete mithaḥ samāḥ ।

On the upper portion [of the cube], in the southeast corner (*agnikoṇa*), lies the cube corresponding to the bigger segment, below which there lies one cuboid, another is northward, and the other one is towards west. All these are equal to each other.

It can also be easily observed from Fig. 4B that the SET block would be a cube with all sides equal to b and its volume would evidently be the largest of all. Hence Nīlakaṇṭha uses the epithet "*mahato ghanaḥ*" to refer to that in verse 6. The two blocks that are adjacent to this cube in the same plane would be NET and SWT and their height will be equal to that of SET, which is b. And the block SEB that is right below SET has its height equal to a. Here, again Nīlakaṇṭha briefly indicates that they are equal to one another in their volumes which will be explained in the following verses.

6.3.4 Properties of Blocks with Unequal Edges

षडेते नैव[9] खण्डाः स्युः समद्वादशबाहवः ॥7॥
खण्डयोः समताभावात् तत्समत्वे समा भुजाः ।

ṣaḍete naiva khaṇḍāḥ syuḥ samadvādaśabāhavaḥ ॥7॥
khaṇḍayoḥ samatābhāvāt tatsamatve samā bhujāḥ ।

These six blocks will not be having twelve equal edges since the segments [marked to dissect the cube] are

[9] In the only edition of the text that is currently available, the first two words have been clubbed together and printed as षडेतेनैव। Such a reading could thoroughly confuse the readers as they may be tempted to split the word षट्+एतेन+एव, which would lead to completely different meaning that does not make any sense in the present context.

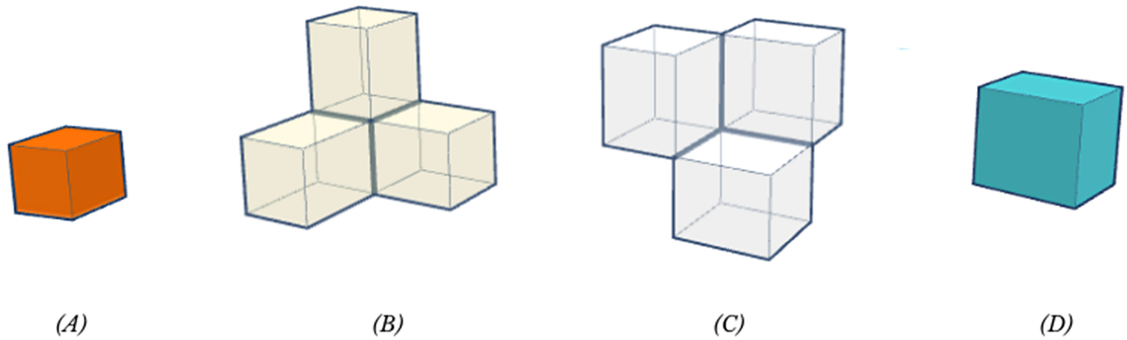

Fig. 5 Dissected blocks that add up to form the cube

unequal. Had they (segments) been equal, the edges would also be equal.

विषमे द्वादशाश्रेऽपि पार्श्वयोस्तु मिथः समम् ॥8॥
फलमूर्ध्वमधश्चापि षट्सु पृष्ठफलेषु तु ।
मिथः प्रतिदिशोस्तुल्यं त्रिविधं स्यात्फलं ततः ॥9॥

viṣame dvādaśāśre'pi pārśvayostu mithaḥ samam ॥8॥
phalamūrdhvamadhaścāpi ṣaṭsu pṛṣṭhaphaleṣu tu ।
mithaḥ pratidiśostulyaṃ trividhaṃ syātphalaṃ tataḥ ॥9॥

Even in a block with twelve unequal edges, the area (*phala*) corresponding to two mutually opposite sides (*pārśva*) is indeed equal. [The areas of] the top and bottom surfaces will also be equal. Since among the six surface areas, the two corresponding to opposite sides are equal, there are only three variant areas.

Having described the nature of six cuboids in one verse, Nīlakaṇṭha then proceeds to make his observations about the surface area of the different faces of these six cuboids formed as a result of dissecting the original cube. It may also be mentioned here, that in the first of above verses the terms *bhuja* and *bahu* which are generally used to denote the sides of a two-dimensional figure such as triangle, square, etc., have been employed for denoting edges. It is clear from this construction that blocks adjacent to the smaller cube — NEB, SWB and NWT, and the blocks adjacent to the bigger cube —NET, SWT and SEB, are all cuboids. Since these blocks are not cubes, Nīlakaṇṭha makes a pertinent observation with regard to their sides in general, as it will be useful later. He notes that in a generic cuboid, though the edges are not equal, the area of any two opposite sides are equal.

6.3.5 General prescription for the computation of the surface area and volume

विस्तारायामपिण्डेषु वध एव द्वयोर्द्वयोः ।
विस्तारायामयोर्घात उपरिष्टात्तलेऽपि च ॥10॥

vistārāyāmapiṇḍeṣu vadha eva dvayordvayoḥ ।
vistārāyāmayorghāta upariṣṭāttale'pi ca ॥10॥

Among the [three quantities] length (*āyāma*), breadth (*vistāra*) and height (*piṇḍa*), the product of any two indeed [give the surface areas]. In the case of top and bottom surfaces, the product of length and breadth [gives the area].

विस्तारोच्छ्रितिघातस्स्यात् ह्रस्वयोः पार्श्वयोर्द्वयोः ।
आयामोच्छ्रितिघातस्स्यात् दीर्घयोः पार्श्वयोर्द्वयोः॥11॥

vistārocchritighātassyāt hrasvayoḥ pārśvayordvayoḥ ।
āyāmocchritighātassyāt dīrghayoḥ pārśvayordvayoḥ ॥11॥

The product of breadth and height would be the area of two smaller upright sides. The product of length and height would be the area of two bigger upright sides.

त्रिष्वेकमितरेणापि हतं घनफलं भवेत् ।

triṣvekamitareṇāpi hataṃ ghanaphalaṃ bhavet ।

Considering any one of the three [areas], multiplying it by the other [quantity which is not involved in the generation of the area] gives the volume (*ghanaphala*) of the block.

Having said that there are six blocks which are not perfect cubes and also outlining how to find the surface area, Nīlakaṇṭha now enunciates how to find the volumes of these blocks. He asks us to find the area of one of the sides, and multiply that by the third dimension. This is a general prescription to find the volume of a cuboid. Following this, Nīlakaṇṭha goes into deducing the volume of the groups of three blocks adjacent to the smaller and bigger cube. What is noteworthy here is his systematic and lucid explanation, that would enable the student to appropriate the validity of an algebraic identity through a logical stream of thinking (Fig. 5).

		$\bar{1}$	$\overset{\vert}{9}$	$\bar{6}$	$\bar{8}$	$\overset{\vert}{3}$	*Root*	Steps
Subtraction : 2^3	$-$		8				2	$-a^3$
Division : 3×2^2	12)	1	1	6	(7			
	$-$		8	4			27	$-3a^2b$
			3	2	8			
Subtraction : $3 \times 2 \times 7^2$	$-$		2	9	4			$-3ab^2$
				3	4	3		
Subtraction : 7^3	$-$			3	4	3		$-b^3$
						0		
Cube root	=						**27**	

Fig. 6 Algorithm of finding the cube-root

6.3.6 Volume of the blocks attached to the smaller cube

तदत्र त्रिषु तुल्येषु पूर्वोक्तेषु घनाप्तये ॥12॥
महता हन्यतेऽल्पस्य वर्गः खण्डस्य च त्रिषु ।

tadatra triṣu tulyeṣu pūrvokteṣu ghanāptaye ॥12॥
mahatā hanyate'lpasya vargaḥ khaṇḍasya ca triṣu ।

In order to obtain the volume of the three identical [blocks] mentioned above [i.e., the ones adjacent to the smaller cube], in all the three instances, the square of the smaller segment is multiplied by the bigger segment.

अल्पखण्डघनेनैषां सह सन्धीयते तु यः ॥13॥
भागस्तत्फलमल्पस्य वर्गतुल्यं यतस्ततः ।
महता हन्यते तत्तद्घनात्मकफलाप्तये ॥14॥

alpakhaṇḍaghanenaiṣāṃ saha sandhīyate tu yaḥ ॥13॥
bhāgastatphalamalpasya vargatulyaṃ yatastataḥ ।
mahatā hanyate tattadghanātmakaphalāptaye ॥14॥

Since the area of the side of these [three blocks] that is attached to the smaller cube is equal to the square of the small segment, in order to obtain the volume of that [block], that [area] is therefore multiplied by the bigger segment.

Since the blocks NEB, SWB and NWT share one of their sides with NWB, the perfect cube of the small segment, the area of the side it shares with the small cube is clearly a^2. In all these three blocks, the dimension perpendicular to the shared side, is b. Hence the *ghanaphala* or volume of these three cubes is equal to a^2b.

6.3.7 Volume of the blocks attached to the bigger cube

महतश्च घनेनैभिः सन्धीयन्ते त्रयोऽपि ये ।
पिण्डेऽल्पखण्डतुल्यास्ते विस्तारायामयोः पुनः ॥15॥
खण्डेन महता तुल्याः तद्वर्गेऽल्पहते ततः ।
प्रत्येकं स्यात् फलं तेषां त्रिघ्नं समुदितं भवेत् ॥16॥

mahataśca ghanenaibhiḥ sandhīyante trayo'pi ye ।
piṇḍe'lpakhaṇḍatulyāste vistārāyāmayoḥ punaḥ ॥15॥
khaṇḍena mahatā tulyāḥ tadvarge'lpahate tataḥ ।
pratyekaṃ syāt phalaṃ teṣāṃ trighnaṃ samuditaṃ bhavet ॥16॥

Similarly, those three blocks which are attached with these [aforesaid three blocks] as well as the bigger cube, will have thickness equal to the smaller segment and the other two dimensions equal to the bigger segment. Therefore the square of that [bigger segment] multiplied by the smaller segment would be the volume of each one. That multiplied by three would be the combined volume of those [three blocks which are adjacent to the bigger cube].

The blocks NET, SWT and SEB share one of their sides with SET, the perfect cube of the bigger segment. The area of the side it shares with the bigger cube is clearly b^2. In all these three blocks adjacent to the big cube, the dimension perpendicular to the shared side, be it length (*āyāma*), breadth (*vistāra*) and height (*piṇḍa*), is a. Hence the *ghanaphala* or volume of these three cubes is equal to ab^2.

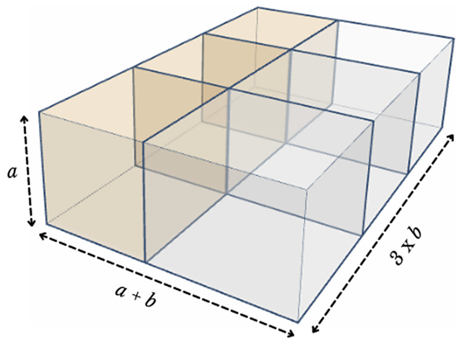

Fig. 7 Blocks which are not perfect cubes can be rearranged to get a combined volume of $3(a+b)ab$

6.3.8 Combined volume of the eight blocks

एवं द्वेधा विभागोऽत्र टष्सु चैकीकृते त्रिक ।
वर्गौ त्र्यन्यहतौ खण्डघनौ यौ तद्युतिर्घनः ॥17॥

evaṃ dvedhā vibhāgo'tra ṣaṭsu caikīkṛte trike ।
vargau tryanyahatau khaṇḍaghanau yau tadyutirghanaḥ ॥17॥

Thus, here, there are two kinds [of blocks]. Among the six [blocks], when two trios are combined, we get the squares [of these two segments independently] multiplied by 3 and the other segment. [Along with these] when the sum of cubes [of the measures of two segments] are added, gives the volume [of the undissected solid block].

घनयुक्त्युपयोगी स्यादेष खण्डघनस्त्विह ।

ghanayuktyupayogī syādeṣa khaṇḍaghanastviha ।

This dissection of a block is indeed useful for showing the rationale of the cubing method (algebraic identity of a cube).

Thus by employing an elegant demonstration of dissecting a cube into eight parts by three similar cuts, Nīlakaṇṭha has effectively shown that the volume of the eight blocks put together exactly gives rise to the RHS of the algebraic identity given in (1). Based on the procedure given for the cube-root extraction given in *Āryabhaṭīya*, it is evident that this expression has been ingeniously used by Indian mathematicians since the times of Āryabhaṭa.

Nīlakaṇṭha does not simply rest there. He playfully rearranges these cubes, to arrive at another equation which is a smart rearrangement of the terms in the aforesaid identity.

6.3.9 Employing this identity in the extraction of cube-root

In fact, we know that mathematically the process of finding the cube of a number and extracting its cube-root are indeed

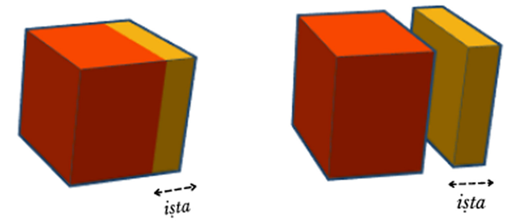

Fig. 8 Cutting the cube at a given point

mutually inverse procedures. Clearly realizing this, Āryabhaṭa in his *Āryabhaṭīya* has prescribed an algorithm for the extraction of cube-root, as mentioned in the 6.1. The procedure outlined by Āryabhaṭa employing this algebraic identity demonstrated by Nīlakaṇṭha is best illustrated with a simple example of finding the cube-root of the number 19683 in Fig. 6.

6.3.10 Rearranging the blocks to illustrate algebraic identity (1) in another form

It was shown earlier that the dissection of the cube has resulted in six cuboid blocks, which are of two kinds. The blocks adjacent to the smaller cube —NEB, SWB and NWT belong to one variety and each of them produce the volume a^2b. The blocks adjacent to the bigger cube —NET, SWT and SEB, have the volume ab^2. Here, Nīlakaṇṭha prescribes to conjoin one each of the first set with one from the second set in order to prove (4).

खण्डाभ्यां वा हतो राशिः त्रिघ्नः खण्डघनैक्ययुक् ॥18॥

khaṇḍābhyāṃ vā hato rāśiḥ trighnaḥ khaṇḍaghanaikyayuk ॥18॥

Or, the [given] number (*rāśi*) multiplied by [its] two components and by three, added with the cubes of those components [gives the cube of the given number].

इत्येतद्युक्तयेऽप्यत्र तुल्ययोस्त्रिकयोर्द्वयोः ।
एकैकं पृथगादाय संश्लिष्टे यत् त्रिकद्वयम् ॥19॥
अल्पखण्डसमं पिण्डे विस्तारे महता समम् ।
कृत्स्नेन राशिना तुल्यम् आयामे तत्त्रयं त्विह ॥20॥
अल्पखण्डहतो राशिः भूयोऽपि महता हतः ।
त्रिघ्नश्च स्याद्घनैक्यञ्च भवेदष्टासु च द्वयम् ॥21॥

ityetadyuktaye'pyatra tulyayostrikayordvayoḥ ।
ekaikaṃ pṛthagādāya saṃśliṣṭe yat trikadvayam ॥19॥
alpakhaṇḍasamaṃ piṇḍe vistāre mahatā samam ।
kṛtsnena rāśinā tulyam āyāme tattrayaṃ tviha ॥20॥
alpakhaṇḍahato rāśiḥ bhūyo'pi mahatā hataḥ ।
trighnaśca syādghanaikyañca bhavedaṣṭāsu ca dvayam ॥21॥

Even for giving the rationale of the above [algebraic expression], taking out one each from the two sets

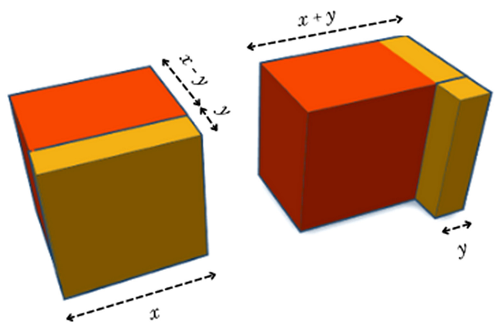

Fig. 9 Rearranged block

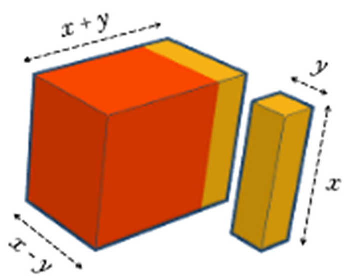

Fig. 10 Volume by parts

of three identical blocks and upon combining them, the resulting two sets of paired three blocks will have the smaller segment as height, the bigger segment as breadth and the unbroken number (sum of the two segments) as the length. This indeed becomes three [newly combined] blocks.

The unequal cuboids can be paired and conjoined along the side where both have the area (*phala*) as $a \times b$ (see Fig. 7). The resulting three new cuboids will have one of its dimensions as a, other as b and the third as $a + b$.

If the given number is N, and it is expressed as the sum of two components a and b, then, the verse essentially gives the following algebraic expression:

$$\begin{aligned} N^3 &= (a+b)^3 \\ &= a^3 + b^3 + 3 \times (a+b) \times a \times b. \end{aligned} \quad (4)$$

So, this result is essentially a rearrangement of the algebraic identity given in (1).

7 Dissection proof of yet another algebraic identity

Having presented the dissection proof of the algebraic identity emplued since the time of Āryabhaṭa, in extracting the cube-root of a number, Nīlakaṇṭha moves on to describe the dissection proof of another interesting algebraic identity which can be extensively made use of to simplify the arithmetic involved in determining the cube of a given number in specific instances. The identity under consideration in given in the following verse:

इष्टोनयुग्राशिवधः वेष्टवर्गघ्नराशियुक् ।
इति द्वेधा विभक्तेऽत्र क्षेत्रे युक्तिः स्फुरेद् घने ॥22॥

iṣṭonayugrāśivadhaḥ veṣṭavargaghnarāśiyuk ।
iti dvedhā vibhakte'tra kṣetre yuktiḥ sphured ghane ॥22॥

Or, by expressing the cube of a given number [*ghana*] in two parts as the product of [the three quantities]—the given number, and the ones obtained by subtracting and adding a desired number [to the given number]—added by the product of the square of the desired number and the given number. The rationale [for this] would be strikingly evident [*sphuret*].

Let x be the number whose cube is to be determined. Let y be *iṣṭa*, a number (such that $x > y$) of one's own choice that could be added or subtracted from x. Then the first half of the above verse essentially gives the RHS of the following algebraic identity (Fig. 8):

$$x^3 = x(x-y)(x+y) + xy^2. \quad (5)$$

This identity is proved through another geometric demonstration by Nīlakaṇṭha.

7.1 Hands-on demonstration of the identity by dissection method

इष्टभागे विदार्यैतं खण्डमादाय योजयेत् ।
शिष्टेनेष्टोनतुल्येऽस्य पार्श्वयोः क्वचिदेव च ॥23॥

iṣṭabhāge vidāryaitaṃ khaṇḍamādāya yojayet ।
śiṣṭeneṣṭonatulye'sya pārśvayoḥ kvacideva ca ॥23॥

Having dissected [a cube] it at any desired portion [i.e., length along one of its sides], the slice that is removed is to be conjoined with the remaining cuboid along one of the adjacent [upright] sides whose measure has been reduced [by slicing a desired portion].

7.1.1 Dimensions of the Rearranged Blocks

राशिनेष्टयुतेन स्यात् आयामोऽस्यैकपार्श्वगः ।
विस्तारोऽपीष्टहीनेन राशिनैव समः क्वचित् ॥24॥

rāśineṣṭayutena syāt āyāmo'syaikapārśvagaḥ ।
vistāro'pīṣṭahīnena rāśinaiva samaḥ kvacit ॥24॥

The length (*āyāma*) of one of the sides of this [rearranged block] would be equal to the given number increased by the desired portion and breadth (*vistāra*) on one side (*kvacit*) equal to the given number decreased by the desired portion.

यत्रैष निहितः खण्डस्तत्र स्यान्महता समः ।
विस्तारः शिखरे तस्मिन् खण्डयित्वा पृथक्कृते ॥25॥
इष्टोनराशिना तुल्यो विस्तारस्तद्युतेन च ।
आयामे राशिना पिण्डे कृत्स्नेनैव समो ह्ययम् ॥26॥

yatraiṣa nihitaḥ khaṇḍastatra syānmahatā samaḥ ।
vistāraḥ śikhare tasmin khaṇḍayitvā pṛthakkṛte ॥25॥
iṣṭonarāśinā tulyo vistārastadyutena ca ।
āyāme rāśinā piṇḍe kṛtsnenaiva samo hyayam ॥26॥

[On the side] where the [dissected] slice is placed (*nihitaḥ*), the breadth would be equal to the greater value (i.e., undivided given number). When the protruding part (*śikhare tasmin*) is dissected and separated, the width would become equal to difference between the given number (*rāśi*) and the desired portion (*iṣṭa*), and the length would be its sum with the given number. In its height (*piṇḍa*) it would indeed be equal to the given number.

As per the prescription of Nīlakaṇṭha, from a cube with all sides of measure x, a slice of width y (*iṣṭa*) is to be dissected. Then it is to be attached to anyone of the perpendicular sides other than the one from which it was cut, as well as the side parallel to it as indicated in Fig. 9. It is further noted that the portion of the slice that is protruding has to be dissected. At this stage, the way these blocks will resemble is depicted in Fig. 10. Hence the dimensions of the bigger chunk of the rearranged block would be:

$$\text{length} = (x + y), \text{width} = (x - y), \text{height} = x.$$

In addition to this, there will be one more block which has been obtained by chopping off the protruding part, as seen in 9. In the following verses Nīlakaṇṭha presents the volumes of these two different components that have been obtained by dissecting a cube.

7.1.2 Volumes of the New Blocks

खण्डः पृथक्कृतोऽन्यो यः स च राशिसमोच्छ्रितिः ।
विस्तारायामयोरिष्टतुल्यं घनफलं द्वयोः ॥27॥
इष्टोनयुक्तविस्तारदैर्घ्यो राशिसमोच्छ्रितिः ।
यस्तत्र तद्वधोऽन्यत्र राशिनेष्टकृतिर्हता ॥28॥

khaṇḍaḥ pṛthakkṛto'nyo yaḥ sa ca rāśisamocchritiḥ ।
vistārāyāmayoriṣṭatulyaṃ ghanaphalaṃ dvayoḥ ॥27॥
iṣṭonayuktavistāradairghyo rāśisamocchritiḥ ।
yastatra tadvadho'nyatra rāśineṣṭakṛtirhatā ॥28॥

The block that was separated [by slicing] indeed has the height equal to the given number, and its length and width are equal to the *iṣṭa* [measure of the thickness that was sliced]. Now, for [obtaining] the volume of these two blocks, in one block whose breadth and length are *rāśi* diminished and increased by *iṣṭa* respectively, their product with height equal to *rāśi*, would have its volume, and in the other block, the square of *iṣṭa* multiplied by *rāśi* would give the volume.

एवं क्षेत्रविभागेन घनयुक्तिरिहोदिता ।

evaṃ kṣetravibhāgena ghanayuktirihoditā ।

Thus, in this way, by means of dissecting the blocks, the rationale behind the process of obtaining the cube [of a number] technique has been explained.

In the verse cited above, we come across the word *śikhara*. This word literally means peak. However in this context, it is to be understood as something that is protruding. While delineating the procedure, it has been stated by Nīlakaṇṭha that this portion has to be chopped off. Having done this, we get two chunks whose volumes have to be computed. They are given by:

Volume of the bigger block,

$$\begin{aligned} V_1 &= \text{length} \times \text{width} \times \text{height} \\ &= (x + y) \times (x - y) \times x. \end{aligned} \tag{6}$$

Volume of the small block,

$$V_2 = y \times y \times x = xy^2. \tag{7}$$

Adding (6) and (7), we obtain the volume of the entire cube

$$\begin{aligned} x^3 &= V_1 + V_2 \\ &= x(x - y)(x + y) + xy^2, \end{aligned} \tag{8}$$

which is the same as the RHS of (5), thereby proving the algebraic identity.

8 Conclusion

It was shown in this paper that Nīlakaṇṭha has provided rationale for three algebraic expressions by resorting to an elegant dissection proof. These ingenious proofs or *upapattis* stand testimonial to the unique and novel pedagogical approach adopted by Indian mathematicians in order to understand the validity of a mathematical result.

Visual representation is a very useful and powerful way of communicating abstract mathematical concepts with the students. Using models and manipulatives enable learners to make connections between their own experience and the mathematical concepts that they learn from textbooks. It is particularly for this reason that such approaches have been strongly advocated and emphasized in recent times (Larbi & Mavis, 2016).

Even those students who are comfortable with arithmetic, face problems when it comes to dealing with algebra. Remembering algebraic identities becomes far more difficult for students who are not that mathematically inclined and even generates a phobia in their minds (Ojose, 2011). It is here that visual representations and do-it-yourself (DIY) techniques come in handy to facilitate students to recall and apply their knowledge rapidly and accurately to a variety of practical problems.

Nīlakaṇṭha's *Āryabhaṭīyabhāṣya* is especially a glowing example replete with such ingenious demonstrations for various mathematical principles and results. In light of the above, it is clear that the study of commentaries with *upapattis* can aid modern pedagogy, in addition to shining light on the workings of the minds of mathematicians of that age.

Acknowledgements One of the authors Dr. K. Mahesh would like to place on record his sincere gratitude to the project on History of Mathematics in India (HoMI), IIT Gandhinagar, which generously funded this study. The other authors would like to acknowledge MHRD for the generous support extended to them to carry out research activities on Indian science and technology by way of initiating the Science and Heritage Initiative (SandHI) at IIT Bombay. The authors are also thankful for the valuable inputs from the referee of this journal.

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