FRACTIONAL PARTS OF ĀRYABHAṬA'S SINES AND CERTAIN RULES FOUND IN GOVINDASVĀMI'S BHĀṢYA ON THE MAHĀBHĀSKARĪYA

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The commentary of Govindasvāmi (*circa* A.D. 800–850) on the *Mahā Bhāskarīya* contains the sexagesimal fractional parts of the 24 tabular Sine-differences given by Āryabhaṭa I (born A.D. 476). These lead to a more accurate table of Sines for the interval of 225 minutes. Thus the last tabular Sine becomes

$$3437+44/60+19/60^2,$$

instead of Āryabhaṭa's 3438,

Besides this improvement of Āryabhaṭa's Sine-table, the paper also deals with some empirical rules given by Govindasvāmi for computing tabular Sine-differences in the argumental range of 60 to 90 degrees. The most important of these rules may be expressed as

$$D_{24-p} = [D_{24}-(1+2+\ldots+p).c/60^2].(2p+1),$$

where

$$p = 1, 2, \ldots, 7;$$

and $D_{17}, D_{18}, \ldots. D_{24}$ are the tabular Sine-differences with D_{24} being given, in the usual mixed sexagesimal notation, as

$$a+b/60+c/60^2.$$

SYMBOLS

$a; b, c$	The usual notation for writing a number with whole part 'a' (say, in minutes) separated from its sexagesimal fractional parts (of various orders), 'b' (in second), 'c' (in thirds), . . . , by a semicolon.
$D_1, D_2, \ldots$	Tabular Sine-differences such that $D_n = R \sin nh - R \sin (n-1)h;\ n = 1, 2, \ldots$
h	Uniform tabular interval.
$L(h)$	Last tabular Sine-difference when the tabular interval is h, so that $L(h) = R - R \cos h$.
m, n, p	Positive integers.
R	Radius, *Sinus totus*, norm.

1. INTRODUCTION

It is well known[1] that the *Āryabhaṭīya* of Āryabhaṭa I (born A.D. 476) contains a set of 24 tabular Sine-differences. In the modern language we can say that the work tabulates, to the nearest whole number, the values of

$$D_n = R \sin nh - R \sin (n-1)h$$

$$\text{for} \qquad n = 1, 2, \ldots\ldots, 24;$$

where the uniform tabular interval h is equal to 225 minutes and the norm R is defined by

$$R = 21600/2\pi \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

Āryabhaṭīya, II, 10 gives[2]

$$\pi = 3{\cdot}1416, \text{ approximately.}$$

Using this approximation of π, the definition (1) gives

$$R = 3437{\cdot}73872, \text{ nearly}$$
$$= 3437;\ 44,\ 19 \text{ to the nearest third.}$$

Thus, to the nearest minute, the 24th tabular Sine (the *Sinus Totus* or the radius) will be given by

$$R = R \sin 90° = 3438.$$

By employing his own peculiar alphabetic system[3] of expressing numbers, Āryabhaṭa could express the 24 tabular Sine-differences just in one couplet which runs as follows:

225 224 222 219 215 210 205 199 (191) 183 174 164
मखि भखि फखि धखि णखि ञखि ङखि हस्झ स्ककि किष्ग श्घकि किघ्व ।
154 143 131 119 106 93 79 65 51 37 22 7
घ्लकि किग्र हक्य धाहा स्त स्ग श्झ ङ्व ल्क प्त फ छ कलार्धज्याः ॥ १० ॥

(*Āryabhaṭīya* I, 10; pp. 16–17)

In Kern's edition (Leiden 1874), which is used here, the text and commentary both give the reading *svaki*, 250 (a wrong value), for the ninth tabular Sine-difference. It is stated by Sen[4] that Fleet pointed out the mistake as early as 1911. However, it must be noted that although the commentary reading is *svaki*, the translation or explanation given by the commentator (Parameśvara, *circa* A.D. 1430) is 'candrāṅkaikaḥ', 191, which is correct. This shows that *svaki* was not the original reading.

In fact, Śaṅkaranārāyaṇa (A.D. 869) in his commentary[5] on *Laghu Bhāskarīya* quotes the above couplet in full with the reading *skaki*, 191 (which is correct), instead of the wrong reading *svaki*, 250. That in the original text of *Āryabhaṭīya* the reading was *skaki* has also been confirmed by consulting the manuscripts[6] of the commentaries of Bhāskara I (A.D. 629)[7] and Sūryadeva Yajva (born A.D. 1191). Hence it is certain that the original reading was *skaki* which is adopted here as well as by other translators.*

* It is now evident that the reading in the commentary by Parameśvara has also been *skaki* originally and not *svaki* as appears in the printed edition.

These tabular Sine-differences are shown in Table I.

Instead of tabulating the Sine-differences to the nearest whole minutes, if they are tabulated up to the second order sexagesimal fraction, then the tabular values should be given in minutes, seconds, and thirds. The sexagesimal fractional parts (seconds and thirds), in defect or in excess, of the Āryabhaṭa's Sine-differences are found stated in the commentary (gloss) of Govindasvāmi (*circa* A.D. 800–850)[8] on the *Mahābhāskarīya* of Bhāskara I (early seventh century A.D.), both belonging to the Āryabhaṭa School of Indian Astronomy. These fractional parts (*avayavāḥ*) are described below in section two of the paper. Certain other rules concerning the computations of Sine-differences, as found in the same commentary, are discussed in the subsequent sections of the paper.

TABLE I

n	Actual value of $R \sin nh$ ($R = 10800/3{\cdot}1416$, and $h = 225$ min.)	Actual Sine-diff. D_n	Āryabhaṭa's Sine-diff.	Govinda-svāmi's fractional parts	Āryabhaṭa's Sine-diff. improved by Govinda-svāmi
1	224; 50, 19, 56	224; 50, 19, 56	225	− 9, 37	224; 50, 23
2	448; 42, 53, 48	223; 52, 33, 52	224	− 7, 30	223; 52, 30
3	670; 40, 10, 24	221; 57, 16, 36	222	− 2, 42	221; 57, 18
4	889; 45, 8, 6	219; 4, 57, 42	219	+ 4, 57	219; 4, 57
5	1105; 1, 29, 37	215; 16, 21, 31	215	+16, 22	215; 16, 22
6	1315; 33, 56, 21	210; 32, 26, 44	210	+32, 26	210; 32, 26
7	1520; 28, 22, 38	204; 54, 26, 17	205	− 5, 34	204; 54, 26
8	1718; 52, 9, 42	198; 23, 47, 4	199	−36, 12	198; 23, 48
9	1909; 54, 19, 5	191; 2, 9, 23	191	+ 2, 09	191; 2, 09
10	2092; 45, 45, 51	182; 51, 26, 46	183	− 8, 33	182; 51, 27
11	2266; 39, 31, 6	173; 53, 45, 15	174	− 7, 02	173; 52, 58
12	2430; 50, 54, 6	164; 11, 23, 0	164	+12, 10	164; 12, 10
13	2584; 37, 43, 44	153; 46, 49, 38	154	−13, 11	153; 46, 49
14	2727; 20, 29, 23	142; 42, 45, 39	143	−17, 14	142; 42, 46
15	2858; 22, 31, 0	131; 2, 1, 37	131	+ 2, 02	131; 2, 02
16	2977; 10, 8, 37	118; 47, 37, 37	119	−12, 22	118; 47, 38
17	3083; 12, 50, 56	106; 2, 42, 19	106	+ 2, 42	106; 2, 42
18	3176; 3, 23, 11	92; 50, 32, 15	93	− 9, 28	92; 50, 32
19	3255; 17, 54, 8	79; 14, 30, 57	79	+14, 31	79; 14, 31
20	3320; 36, 2, 12	65; 18, 8, 4	65	+18, 08	65; 18, 08
21	3371; 41, 0, 43	51; 4, 58, 31	51	+ 4, 59	51; 4, 59
22	3408; 19, 42, 12	36; 38, 41, 29	37	−21, 19	36; 38, 41
23	3430; 22, 41, 43	22; 2, 59, 31	22	+ 3, 00	22; 3, 00
24	3437; 44, 19, 23	7; 21, 37, 40	7	+21, 37	7; 21, 37

2. FRACTIONAL PARTS OF ĀRYABHAṬA'S SINE-DIFFERENCES

Described in the usual Indian word-numerals (Bhūtasaṅkyās), the seconds and thirds (in defect or in excess) of all the 24 Āryabhaṭa's Sine-differences appear on page 200 of the printed edition (Madras, 1957) of Govindasvāmi's commentary on the *Mahābhāskarīya*. They are as follows (the first two digits in each figure-group *of the text* denote the thirds):

9,37	7,30	2,42	4,57
सप्ताग्निरन्ध्राणि,	वियद्गुणागं,	नेत्राब्धिनेत्रं,	मुनिपञ्चवेदाः ।
16,22	32,26	5,34	36,12
द्वयक्ष्यष्टयः,	षण्णयनद्विरामा,	वेदाग्निभूतं,	रविषट्कृशानुः ॥
2,09	8,33	7,02	12,10
रन्ध्राभ्रपक्षं,	गुणपावकाष्टौ,	चक्षुर्वियत्सप्त,	खचन्द्रसूर्याः ।
13,11	17,14	2,02	12,22
रुद्राग्निचन्द्रा,	मनुसप्तसोमा,	दस्राभ्रनेत्रं,	नयनंद्विसूर्यम् ॥
2,42	9,28	14,31	18,08
अक्ष्यब्धिपक्षं,	वसुनेत्ररन्ध्रं,	चन्द्राग्निविद्या,	वसुखाष्टचन्द्रम् ।
4,59	21,19	3,00	21,37
रन्ध्रेषुवेदं,	नवरूपमिध्मं,	खाभ्राग्नयस्,	सप्तगुणेध्मसंख्यम् ॥

(Govindasvāmi's commentary on the *Mahābhāskarīya* under IV, 22).

After describing these values the commentary says (p. 201):

इत्युक्तास्तत्पराद्याः स्युरेते हीनाधिकांशकाः ।
गुणानां ते ततः शोध्या मख्यादौ योजिता अपि ॥
त्रि-त्रि-द्वि-रूप-नेत्रै-क-द्वि-चन्द्रै-के-न्दु-संख्यया ।
एक-त्रि-रूप-नेत्रै-श्च ज्याविद्भिगणकैः क्रमात् ॥

'These are the fractional parts, thirds first, in defect or in excess, of the Sine-differences. They are subtracted from, and added to, (the Āryabhaṭa's Sine-differences) *makhi*, etc., by the calculators expert in Sines (taking) 3, 3, 2, 1, 2, 1, 2, 1, 1, 1, 1, 3, 1, 2, in succession (from the set)'.

These fractional parts with their proper signs are tabulated in TABLE I. The resulting tabular Sine-differences are also given in the table along with the actual values for the purpose of comparison.

3. AN APPROXIMATE RULE CONCERNING THE LAST TABULAR SINE-DIFFERENCE

For finding an approximate value of the last Sine-difference with tabular interval $h/2$, from the last Sine-difference when the tabular interval is h, the commentary (p. 199) of Govindasvāmi on the *Mahābhāskarīya* gives a simple rule as follows:

अन्त्यगुणस्य तावत् चतुर्भागः, तदर्धकाष्ठान्त्यज्या

'The fourth part of the last (tabular) Sine-difference (corresponding to a tabular interval of arc h) is the last (tabular) Sine-difference corresponding to half of the (given tabular) arc.'

That is,

$$(1/4).\ L(h) = L(h/2).$$

The work gives the following illustrations of the rule:

$$(1/4).\ L(450) = L(225),$$
$$(1/4).\ L(225) = L(112;\ 30)$$
$$(1/4).\ L(112;\ 30) = L(56;\ 15)$$

'In this way', says the author, 'the last tabular Sine-difference corresponding to any tabular arc (of the type $h/2^n$) should be obtained. Thus we have the rule

$$L(h/2^n) = L(h)/4^n.$$

Rationale: We have

$$L(h/2) = R-R\cos(h/2) = 2\ R\sin^2(h/4).$$

Now

$$\begin{aligned} L(h) &= R-R\cos h = 2\ R\sin^2(h/2) \\ &= 8\ R\sin^2(h/4).\ \cos^2(h/4) \\ &= 4.\ L(h/2).\ \cos^2(h/4), \text{ by the above.} \end{aligned}$$

Therefore,

$$\begin{aligned} L(h/2) &= (1/4).\ L(h).\ \sec^2(h/4) \\ &= (1/4).\ L(h)+(1/4).\ L(h).\ \tan^2(h/4). \end{aligned}$$

From this the rule follows, since (when h is small)

$$\begin{aligned} &(1/4).\ L(h).\ \tan^2(h/4) \\ &\quad = (1/4).\ 2R\sin^2(h/2).\ \tan^2(h/4) \\ &\quad = h^4/128R^3, \text{ approximately,} \end{aligned}$$

which is negligible.

For an alternative rationale *see* Section 4 below.

4. A Crude Rule for Computing Tabular Sine-differences in the Third Sign (60° to 90°)

After giving the method of finding the last tabular Sine-difference D_n (described in the last section), the commentary (p. 199) of Govindasvāmi on *Mahābhāskarīya* gives the following crude rule for obtaining the other tabular Sine-differences (lying in the third sign only) from D_n.

सा पुनस्त्र्यादिविषमसंख्यागुणिता तदधःप्रभृत्युत्क्रमतस्तद्भागज्या । एवं तृतीयराशिज्या-
कल्पना ।

'That (that is, the last tabular Sine-difference) severally multiplied by the odd numbers 3, etc., become the Sine-difference below that (that is, the last-but-one), etc. (that is, the other Sine-differences), counted in the reversed order. This is the method of getting Sine-differences in the third sign.'
That is, from

$$L(h) = D_n,$$

we get

$$3 \times L(h) = D_{n-1},$$

$$5 \times L(h) = D_{n-2},$$

$$\cdots\cdots\cdots\cdots$$

$$(2p+1)\ L(h) = D_{n-p};\ p = 0, 1, 2, \ldots$$

Rationale: We have

$$\begin{aligned} D_{n-p} &= R \sin (n-p)h - R \sin (n-p-1)h \\ &= R \cos ph - R \cos (p+1)h, \text{ as } nh = 90^\circ, \\ &= 2R \sin (h/2).\ \sin (ph+h/2) \\ &= 2R \sin^2 (h/2).\ \frac{\sin (ph+h/2)}{\sin (h/2)} \\ &= D_n.\ \frac{\sin \{(2p+1)h/2\}}{\sin (h/2)} \\ &= (2p+1).\ D_n, \text{ roughly,} \end{aligned}$$

since h ($= 90/n$ degrees) is small and $(ph+h/2)$ is less than 30 degrees in the third sign. Thus follows the above crude rule.

From this rule it is clear that

$$D_{n-1} = 3D_n$$

$$D_{n-2} = 5D_n$$

$$D_{n-3} = 7D_n, \text{ etc.}$$

Now

$$\begin{aligned} L(h) &= D_n \\ L(2h) &= D_n + D_{n-1} = (1+3)D_n \\ &= 4L(h) \\ L(4h) &= D_n + D_{n-1} + D_{n-2} + D_{n-3} \\ &= (1+3+5+7)D_n \\ &= 4^2 L(h). \end{aligned}$$

Thus, in general, we have

$$L(2^n h) = 4^n L(h),$$

or

$$L(h) = L(2^n h)/4^n$$

which is equivalent to the rule described in section 3 above.

It can be easily seen that the rule, although simple, is very gross. The D_{24}, of TABLE I, when multiplied by 3, 5, 7, ..., 15, will not give results equal to D_{23}, D_{22}, D_{21}, ..., D_{17}, respectively. 'This is no fault, as the manipulation is not complete', says Govindasvāmi. He, therefore, gives a modification of this rule which we describe now.

5. GOVINDASVĀMI'S MODIFIED RULE FOR COMPUTING TABULAR SINE-DIFFERENCES IN THE THIRD SIGN

In the commentary (p. 201) of Govindasvāmi on the *Mahābhāskarīya* is found an excellent rule for computing, from a given last tabular Sine-difference D_n, the other Sine-differences lying in the third sign (60 degrees to 90 degrees). The text says:

अन्त्यज्या तावदेकादिसंकलितगुणिततत्पराहीना त्र्यादिविषमगुणिता फादितुल्याऽन्त्यभवने भवेदिति ।

'Diminish the last (tabular) Sine-difference by its thirds multiplied (severally) by the sums of (the natural numbers) 1, etc. The results (so obtained) multiplied by the odd number 3, etc., become the (tabular) Sine-differences, in the third sign, starting from "*pha*" (that is, the last-but-one Sine-difference).' That is, taking the last Sine-difference

$$D_n = a + b/60 + c/60^2 \text{ minutes}$$
$$= a;\ b,\ c \text{ say,}$$

we have

$$D_{n-1} = [D_n - 1 \times c/60^2] \times 3$$
$$D_{n-2} = [D_n - (1+2)c/60^2] \times 5$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$D_{n-p} = [D_n - (1+2+\ldots+p)c/60^2].\ (2p+1),$$
$$p = 0, 1, 2, \ldots$$

Illustration: We take, for the last Sine-difference, the value

$$D_{24} = 7;\ 21,\ 37$$

as found in the work itself (*see* TABLE I). Applying the above rule, we get

$$D_{23} = (D_{24} - 1 \times 37/60^2) \times 3$$
$$= 22;\ 3,\ 0.$$
$$D_{22} = [D_{24} - (1+2) \times 37/60^2] \times 5$$
$$= 36;\ 38, 50.$$

Similarly all the differences up to D_{17} may be worked out. These are shown in TABLE II and may be compared with the set of values given in the work itself.

TABLE II

p	D_{n-p} ($n = 24$)	Sine-diff. by the Rule applied to $D_n = 7; 21, 37$	Sine-diff. as given in the work	Actual value	By the Rule applied to $D_n = 7; 21, 38$
0	D_{24}	7; 21, 37	7; 21, 37	7; 21, 38	7; 21, 38
1	D_{23}	22; 3, 0	22; 3, 0	22; 3, 0	22; 3, 0
2	D_{22}	36; 38, 50	36; 38, 41	36; 38, 41	36; 38, 40
3	D_{21}	51; 5, 25	51; 4, 59	51; 4, 59	51; 4, 50
4	D_{20}	65; 19, 3	65; 18, 8	65; 18, 8	65; 17, 42
5	D_{19}	79; 16, 2	79; 14, 31	79; 14, 31	79; 13, 28
6	D_{18}	92; 52, 40	92; 50, 32	92; 50, 32	92; 48, 20
7	D_{17}	106; 5, 15	106; 2, 42	106; 2, 42	105; 58, 30

Rationale: We have already shown (*see* section 4) that

$$D_{n-p} = D_n . [\sin \{(2p+1)h/2\}]/\sin (h/2).$$

Now it is known that[9]

$$\sin m\theta = m \sin \theta - \frac{m(m^2-1^2)}{3!} \sin^3\theta + \frac{m(m^2-1^2)(m^2-3^2)}{5!} \sin^5\theta - \dots$$

Taking in this,

$$m = 2p+1, \text{ and } \theta = h/2$$

we get

$$[\sin \{(2p+1)h/2\}]/\sin (h/2) = (2p+1)-(2/3)p(p+1)(2p+1) \sin^2 (h/2) + f(p) \sin^4 (h/2) - \dots$$

Using this we get

$$\begin{aligned} D_{n-p} &= (2p+1)D_n - D_n . (4/3)(2p+1)(1+2+\dots+p) \sin^2 (h/2) + \dots \\ &= [D_n-(4/3)(1+2+\dots+p)D_n . \sin^2 (h/2)] . (2p+1), \end{aligned}$$

neglecting higher terms which are comparatively small.
This we can write as

$$D_{n-p} = [D_n-(1+2+\dots+p)k] . (2p+1),$$

where

$$\begin{aligned} k &= (4/3) \sin^2 (h/2) . D_n \\ &= (2/3R)D_n^2, \text{ or, } (8R/3) \sin^4 (h/2). \end{aligned}$$

since

$$D_n = R(1-\cos h) = 2R \sin^2 (h/2),$$

Now, in our case,

$$h = 225 \text{ minutes},$$

$$R = 10800/3{\cdot}1416.$$

Hence we easily get

$$k = 1/95{\cdot}2, \text{ nearly.}$$

The numerical value implied in the rule given by Govindasvāmi is

$$= 37/60^2$$

$$= 1/97{\cdot}3, \text{ nearly.}$$

This is quite comparable to the actual value calculated above.

Acknowledgement

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References and Notes

1 The subject of the *Āryabhaṭīya* Sine-differences has been dealt by many previous scholars. Some references are:

(i) Ayyangar, A. A. K.: 'The Hindu Sine-Table'. *Journal of the Indian Mathematical Society*, Vol. 15 (1924–25), first part, pp. 121–26.

(ii) Sengupta, P. C.: 'The Āryabhaṭīyam' (An English Trans.) *Journal of the Department of Letters* (Calcutta University), Vol. 16 (1927), pp. 1–56.

(iii) Sen, S. N.: 'Āryabhaṭa's Mathematics'. *Bulletin of the National Institute of Sciences of India*, No. 21 (1963), pp. 297–319.

2 *The Āryabhaṭīya* with the commentary *Bhaṭadīpikā* of Paramādīśvara (Parameśvara); edited by H. Kern, Leiden, 1874, p. 25. In our paper the page-references to *Āryabhaṭīya* and Parameśvara's commentary on it are according to this printed edition.

3 For an exposition of his alphabetic system of numerals *see*, for example, *History of Hindu Mathematics: A Source Book* by B. Datta and A. N. Singh; Asia Publishing House, Bombay, 1962; pp. 64–69 of part I.

4 Sen, S. N.: 'Āryabhaṭa's Math.' Op. cit., p. 305.

5 *Laghu Bhāskarīya* with the commentary of Śaṅkaranārāyaṇa edited by P. K. N. Pillai; Trivandrum, 1949; p. 17.

6 Vide Manuscripts of the commentaries by Bhāskara I, p. 39, and by Sūryadeva Yajva, p. 20, both in the Lucknow University collection.

7 *Laghu Bhāskarīya* edited and translated by K. S. Shukla; Lucknow University, Lucknow, 1963; p. xxii.

8 *Mahābhāskarīya* of Bhāskarācārya (Bhāskara I) with the *Bhāṣya* (gloss) of Govindasvāmin and the super-commentary *Siddhāntadīpikā* of Parameśvara edited by T. S. Kuppanna Sastri; Govt. Oriental Manuscripts Library, Madras, 1957; p. XLVII. All page-references to Govindasvāmi's commentary (gloss) are according to this edition.

9 *Higher Trigonometry* by A. R. Majumdar and P. L. Ganguli; Bharti Bhawan, Patna, 1963; p. 128.