SINE OF EIGHTEEN DEGREES IN INDIA UPTO THE EIGHTEENTH CENTURY

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The paper deals with both the statements and derivations of the exact value of the Sine of 18 degrees as found in India upto the first half of the eighteenth century A.D. This exact value, which is equivalent to the modern value, namely

$$\sin 18^\circ = (\sqrt{5}-1)/4,$$

is found, apparently for the first time in India, in the *Jyotpatti* of Bhāskara II (twelfth century A.D.) whose way of arriving at the value, however, is not known to us at present.

A few proofs, presented here, are those which are found in the works of Munīśvara, Kamalākara (both seventeenth century), and Jagannātha (eighteenth century). These proofs are mostly geometrical in nature but analytical treatment in terms of trigonometrical and algebraic steps is also involved. Indian familiarity with some foreign material on the subject is also indicated in this connection.

1. INTRODUCTION

The Greek trigonometry, that is, the one used by Hipparchus (second century B.C.) and Ptolemy (second century A.D.), was based on the functional relationship between the chords of a circle and the central angles they subtend. Indian trigonometry, on the other hand, was based on the so-called Indian trigonometric functions namely, Sine, Cosine, Versed Sine, and Coversed Sine (all written usually with initial capital letters to distinguish them from the corresponding modern functions). The Indian Sine of an arc in a circle was defined to be the length of half the chord of twice the arc. In value, it is equal to $R \sin Z$, where R is the radius of the circle of reference and $\sin Z$ is the modern sine of the angle subtended by the arc at the centre. Thus, the predecessor of the basic modern trigonometric function known as the sine of an angle was born apparently in India[1]. Even the modern European word "sine" has descended from the Hindu name, *jīvā* in Sanskrit[2]. (In Fig. 1, lengths PA, OA and AX represent, respectively, the Sine, Cosine, and Versed Sine of the arc PX).

The *Āryabhaṭīya* ($= AB$) of Āryabhaṭa I (born 476 A.D.) is the earliest extant historical work in which the Indian trigonometry is definitely used, but the invention of the Sine function might be dated earlier.

Exact numerical values of the trigonometric functions associated with simple angles like 30, 45 and 60 degrees follow from the definition when triangles containing the above angles are considered inscribed in the circle of reference. When the Sine (and, therefore, also the Versed Sine) of an angle is known, that of its half can be easily computed, for example, by using the so-called *Kramotkramajyā* Rule, namely

$$(R \sin \tfrac{1}{2}Z)^2 = [(R \sin Z)^2 + (R \text{ vers } Z)^2]/4 \qquad \ldots \quad (1)$$

The derivation of (1) is straight-forward by applying the so-called Pythagorean Theorem to the triangle PAX. Among the various computational methods based on exact identities, the one given by (1) seems to be the oldest and was commonly followed in ancient India. This method is explicitly laid down in the *Pañca-siddhāntikā*, IV, 3–4 of Varāhamihira (sixth century A.D.)[3]. The same method is taken to be implied in the *AB*, II, 11 as indicated in its gloss by Nīlakaṇṭha Somayāji (about 1500 A.D.)[4] and as envisaged by Sengupta[5]. The details are found in the *AB* commentary (A.D. 629) by Bhāskara I who gives due credit to a previous teacher, Prabhākara, for the exposition of the subject[6].

Using the exact functional values associated with 30 or 45 degrees, the knowledge of (1) or similar sub-duplicating formulas, can easily enable one to find the exact values of the Sines of 15 and $22\frac{1}{2}$ degrees. But we cannot find the exact value of the Sine of 18 degrees in this manner. Its determination requires some novelty. Thus the matter concerning the exact value of the Sine of 18 degrees must have a matter of some curiosity (if not that of much practical utility) among the early Indians who were quite concerned with the development of trigonometry.

The present paper contains statements of the exact valve of the Sine of 18 degrees and its proofs as found in some important Indian works dated between the twelfth and the eighteenth centuries of our era.

2. Exact Value of the Sine of 18 Degrees

In India, the exact value of the Sine of 18 degrees is found stated in a small trigonometrical tract called *Jyotpatti* which is attributed to the famous Indian astronomer and mathematician Bhāskara II (twelfth century A.D.). The *Jyotpatti* may be considered either Chapter XIV of, or as an apprndix to, the author's *Siddhānta–Śiromaṇi–Golādhyāya* (= *SSG*) which, together with the corresponding *Graha-gaṇita* part, constitute his standard work on Hindu astronomy. The ninth verse of the *Jyotpatti* is[7]

त्रिज्याकृतीषुघातान्मूलं त्रिज्योनितं चतुर्भक्तम् ।
अष्टादशभागानां जीवा स्पष्टा भवत्येवम् ॥ ९ ॥

Trijyā-kṛtīṣu-ghātān-mūlaṃ trijyonitaṃ catur-bhaktam
Aṣṭādaśa-bhāgānaṃ jīvā spaṣṭā bhavatyevam

'Subtract the radius from the square-root of the product of the radius-square and five, and divide by four; that becomes the true Sine of the eighteen degrees'.

This is,

$$R \sin 18^\circ = (\sqrt{5R^2} - R)/4 \qquad \ldots \quad (2)$$

giving the correct modern value

$$\sin 18^\circ = (\sqrt{5} - 1)/4. \qquad \ldots \quad (3)$$

In place of the negative sign in the numerator of (2), if we have the positive sign, the expression will become equal to the exact value of the Sine of 54 degrees. This fact is clearly noted by Muniśvara in his commentary (A.D. 1638), called *Marīci* (= *MC*)[8], on the *Jyotpatti*.

The author of the *MC* later on (A.D. 1646) expressed both the results in his *Siddhānta-sārvabhauma* (= *SSB*), II, 41 in the following words[9]

त्रिज्यावर्गात् पञ्चनिघ्नात् पदं यत्
त्रिज्योनाढ्यं वेदभक्तं ज्यके स्तः ।
धृत्यंशाब्धीष्वंशयोः................................।। ४१ ।।

Trijyā-vargāt pañca-nighnāt padaṃ yat
Trijyonāḍhyaṃ vedabhaktaṃ jyake staḥ
Dhṛtyaṃśābdhīṣvaṃśayoḥ ....
.... 41

'(Separately) diminish and increase the square-root of five times the radius-square by the radius, and divide by four; (the two results so obtained) are the Sines of eighteen degrees and fifty-four degrees (respectively)......'.

That is,

$$(\sqrt{5R^2} \pm R)/4 = \text{Sin } 18^\circ, \text{ or Sin } 54^\circ \qquad \ldots \quad (4)$$

Kamalākara (A.D. 1658) in his *Siddhānta-tattva-viveka* (= *STV*), III, 98–99 (implicitly), and in his own commentary (= *STVC*) on it, also gives the same expressions[10]. In stating these values, he employs an accurate approximation for the square-root of five namely

$$\begin{aligned}\sqrt{5} &= 2;\ 14,\ 10\\ &= 2 + (14/60) + (10/3600)\\ &= 161/72\\ &= 2{\cdot}2361111\ldots\ldots\end{aligned}$$

which is correct to four decimal places.

Jagannātha (first half of the eighteenth century) has expressed the value of the Sine of 18 degrees as follows[11]

त्रिज्यार्द्धवर्गादिषु संगुणाच्च
मूलं खरामांशक जीवयोनं ।
तदर्द्धकं स्याद् धृतिभागजीवा
. . . . ॥ ११ ॥

Trijyārddha-vargādiṣu saṃguṇācca
Mulaṃ kharāṃśaka jīvayonam
Tadarddhakaṃ syād dhṛtibhāgajīvā
.... 11

'Subtract the Sine of thirty degrees from the square-root of the product of the square of half of the radius and five. Half of that (above result) is the Sine of the eighteen degrees........."

That is

$$R \sin 18^\circ = \tfrac{1}{2}[\sqrt{5(R/2)^2} - R \sin 30^\circ] \qquad \dots \quad (5)$$

which is equivalent to (2), since

$$R \sin 30^\circ = R/2.$$

3. Derivations

The *Jyotpatti* and its author's own accompanying commentary on it do not contain any derivation for the expression (2). Proofs of the formula are found in India much later. Below we give some of the derivations.

(I) An incorrect proof given by Lakṣmīdāsa Miśra (about 1500 A.D.), is found quoted in the *MC* (part I, p. 145). It is substantially as follows :

We have

$$\text{Sin } 90^\circ = R$$

which can be written as

$$\text{Sin } 90^\circ = (\sqrt{25R^2} - R)/4 \qquad \dots \quad (6)$$

Now we see that, when the angle is 90 degrees, the coefficient of R^2 (under the square-root sign) in its Sine is 25. Therefore, when the angle is 18 degrees, this coefficient will be, by the rule of proportional parts (*anupātāt*),

$$25 \times (18/90) = 5.$$

Hence, taking 5 (which is one fifth of 25) as the coefficient (of R^2), in place of 25 in the above expression (6),we get the desired value (2) of the Sine of 18 degrees (which is one fifth of 90 degrees) !

The *MC* states that this is no proof and further points out that, had it been so, we would have (by similar wrong argument)

$$\text{Sin } 36^\circ = (\sqrt{10R^2} - R)/4$$

which is not what it should be.

(II) A geometrical proof, found in the *MC* (p. 144), may be presented as follows :

In Fig. 1, XY is a quadrant of the circle of reference with centre at O and radius

$$OX = OY = R.$$

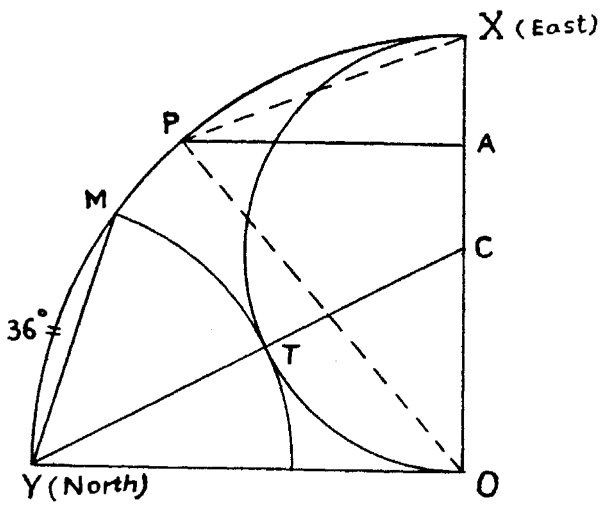

Fig. 1.

A semi-circle is described on OX as a diameter and with centre at C which is the mid-point of OX. Also, with centre at Y and radius YM (where YM is the chord of 36 degrees), another circle is described. This will be found, evidently (*pratyakṣam*), to touch the above semi-circle, the point of contact being T.

Thus,

$$\begin{aligned} \text{Sin } 18^\circ &= YM/2 \\ &= YT/2 \\ &= (YC - TC)/2 \\ &= \tfrac{1}{2}\sqrt{(OY^2+OC^2} - OC/2 \\ &= \tfrac{1}{2}\sqrt{R^2+(R/2)^2} - R/4 \end{aligned}$$

which gives the required expression (2).

It may be pointed out that the *MC* does not give any proof for the tangency of the two circular arcs at T in the above construction, but adds that the proof is dealt in details in a foreign work,

यवनग्रन्थे सविस्तरं प्रतिपादिता ।

'*Yavana-granthe savistaraṃ pratipāditā*',

as it says (*MC*, part I, p. 144).

Same proof is found in the *SSB*, II, 42-48 (A. D. 1646) and author's own commentary on it (part I, pp. 135-137).

(III) Another proof, which is also found in the *MC* (pp. 144–145) on the *Jyotpatti*, is based on the following lemma[12]:

दशास्रभुजवर्गोऽयं भुजत्रिज्यावधेन युक् त्रिज्यावर्गो भवेत् ।

Dasāśra-bhuja-vargo'yaṃ bhuja-trijyā-vadhena yuk trijyāvargo bhavet.

'The square of a side of a regular decagon together with the product of the side and the radius (of the circumscribing circle) is equal to the square of the radius'.

That is,

$$(2x)^2+2x.R = R^2 \qquad \dots \quad (7)$$

where $2x$ is the side of a regular polygon of ten sides inscribed in a circle of radius R, so that x is equal to the Sine of 18 degrees.

The *MC* solves (7) by the usual Indian method[13] of completing the square, that is by multiplying both sides by four and then adding R^2. Thus we get

$$16x^2+8xR+R^2 = 5R^2.$$

Now by taking the square-roots of both sides, we get (omitting the inadmissible negative root)

$$4x+R = \sqrt{5R^2}$$

which yields the desired rule (2).

The above lemma, which is given in the *MC* without proof, is again quoted and proved in *SSB*, II, 49–55 and the author's own commentary on it (part I, pp. 137–141). The algebraic cum trigonometrical details given there are substantially as follows :

Let

$$R \sin 18° = x = R \cos 72°.$$

By using the formula

$$(R/2).R \text{ vers } 2A = (R \sin A)^2$$

we get

$$(R \sin 36°)^2 = (R/2).R \text{ vers } 72°$$
$$= R(R-x)/2.$$

Also, by the same formula, we have

$$R \text{ vers } 36° = (2/R).(R \sin 18°)^2$$
$$= 2x^2/R.$$

Now, by adding the squares of the Sine and Versed Sine of 36 degrees and taking square-root, we get the chord of 36 degrees of arc which is equal to $2x$.

Thus, using the above values

$$(2x)^2 = (R/2).(R-x)+4x^4/R^2$$

or

$$8x^4-x.R^3+R^4 = 8x^2.R^2.$$

This can be written as

$$16x^2.R^2+8x.R^3+R^4 = 64x^4-48x^2.R^2+9R^4$$

or

$$(R^2+4x.R)^2 = (3R^2-8x^2)^2. \qquad \ldots \quad (8)$$

Taking a root, we get

$$R^2+4x.R = 3R^2-8x^2$$

or

$$4x^2+2x.R = R^2$$

the required lemma.

Almost the same derivation is found in the commentary (p. 149) by Kamalākara (1658 A.D.) on his own *STV*, III, 98–101. Moreover, this author also considers the other possible relation obtained by taking square-roots in (8), namely

$$R^2+4x.R = 8x^2-3R^2$$

or

$$2x^2-xR-R^2 = 0.$$

However, even the positive root of this equation, namely

$$x = R$$

is not admissible, since the Sine of 18 degrees should be, as pointed out in the *STVC* (p. 149), less than R.

(IV) Jagannātha's commentary (*SSK*, part II, pp. 1054–55), on the Sanskrit verse which gives the formula (5), contains a different geometrical proof which may be briefly described as follows :[14]

With centre at B, a circle of the desired radius should be drawn on level ground (*see* Fig. 2).* Draw the east-west line and mark the east point K. Mark the point G at 36 degrees (angular distance) from K, Make the full chord KG and join BG. Produce the line GK upto D such that KD is equal to KB. The line segment KD is bisected at H. Join DB.

Here the angle KBG of the smaller triangle is equal to 36 degrees. The measure of each of the angles BKG and BGK is equal to 72 degrees, because the sum of the three angles of a triangle is 180 degrees.

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* The figure and proof (IV) are similar to those found in *Euclid*, XIII, 9 (Heath's edition vol. III, pp. 455-56; Dover, New york, 1956).

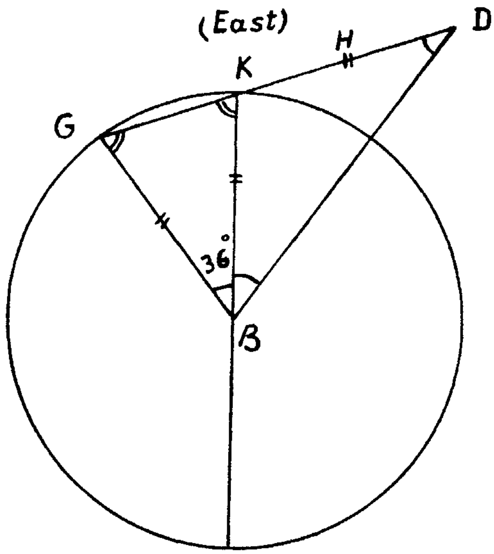

Fig. 2.

Since KB and KD are equal, therefore angle KBD is equal to the angle KDB, whence angle KDB measures 36 degrees. The angle at G being common, the bigger triangle BGD and the smaller triangle KBG are similar.

Suppose the measure of KG is y (*yāvat-tāvat*), so that

$$DG = y+60 \qquad \dots \quad (9)$$

(where the radius BK is taken to be 60).

If (by considering proportions) we get the line BG from the line GD, then how much shall we get from BG. The result is y. Or the ratio GD/BG is equal to the ratio BG/KG. So that

$$GD.y = BG^2 \qquad \dots \quad (10)$$

or, using (9),

$$y^2+60y = BG^2.$$

Adding the square of half the radius, we get

$$y^2+60.y+900 = 0.y^2+0.y+4500$$

or

$$(y+30)^2 = 4500$$

giving

$$HG = (y+30) = 67;4,\ 55,20.$$

From this, we get

$$y = 37;4,55,20$$

which is the full chord of 36 degrees. Half of it will give the Sine of 18 degrees.

So ends the proof.

We note that, for a general radius R (instead of the numerical value 60), this proof amounts to the determination of y from the relation

$$(y+R)/R = R/y$$

which follows from similar triangles in Fig. 2. Also cf. equation (7).

(V) Another geometrical proof is found in the *Sam.Sid.* (Vol. I, pp. 50-51). This is based on constructing a line segment of length x such that it satisfies a relation (between line segments) which is equivalent to

$$(a+x)/a = a/x.$$

where a is equal to $R/2$.

Clearly, x will represent the desired Sine of 18 degrees. This proof, which is an adaptation (with slight modification) of Ptolemy's construction[15] for the chord of 36 degrees, apparently belongs to the Arabic version of the Almagest from which the Sanskrit *Sam.Sid.* was prepared.

References and Notes

[1] See Carl B. Boyer, *A History of Mathematics,* Wiley, New York, 1968, p. 232.

[2] Ibid., p. 278.

[3] The *Pañca-siddhāntikā,* text and translation by David Pingree, pp. 52–53; Royal Danish Academy of Sciences and Letters, Copenhagen, 1970.

[4] The *Āryabhaṭīyam* (with the *Bhāṣya* of Nīlakaṇṭha Somasutvan,) Part I (*Gaṇitapāda*), p. 44; Edited by K. Sambasiva Sastri, Trivandrum, 1930 (Trivandrum Sanskrit Series No. 101).

[5] The *Āryabhaṭīyam* translated into English by P. C. Sengupta, p. 18; *Journal of the Department of Letters* (Calcutta University), Vol. 16 (1927).

[6] Vide the transcript of Bhāskara I's commentary on the *AB* at the Lucknow University, p. 20.

[7] The *Siddhānta-Śiromaṇi* (with the author's commentary), p. 282; Edited by Bapu Deva Sastri and revised by Ganapati Deva Sastri, Chowkhamba, Benaras, 1920 (Kashi Sanskrit Series No. 72).

[8] The *Siddhānta-Śiromaṇi-Golādhyāya* (with the author's own commentary and the *Marīci,* Part I, p. 144; Edited by D. V. Apte, Poona, 1943 (Anandasrama Sansrkit Series No. 122).

The *Jyotpatti* and the *MC* is found here in Chapter V itself, instead of at the end of *SSG*.

[9] The *Siddhānta-sārvabhauma,* Part I, p. 135; Edited by Murlidhara Thakkur, Benaras, 1932 (The Princess of Wales Saraswati Bhavana Text No. 41).

[10] The *Siddhānta-tattva-viveka,* Part I, pp. 148–150; Edited, with the notes of Sudhakara Dvivedi, by Muralidhara Jha, Benares, 1924 (Benares Sanskrit Series No. 1).

[11] The *Samrat-Siddhānta* (= *Sam. Sid*). or the *Siddhānta-sāra-kaustubha* (= *SSK*), Vol. II, p. 1050; Edited by R. S. Sharma and his team, Indian Inst. of Astronomical and Sanskrit Research, New Delhi, Vol. I (1967), Vol. II (1967), Vol. III (1967/68).

Jagannātha's treatment and exposition of the trigonometrical subject (*jyotpatti* or *jyāsādhanā* is found in the portion which is given in the above edition after his Sanskrit translation (called *Sam,. Sid.* or *Sid. Sam*). from an Arabic version, of the *Almagest* in 13 chapters is apparently completed. This portion (which starts with an Invocation and introduction) is written in the usual Hindu style of the Sanskrit verses and is accompanied by author's. own lucid commentary. The portion may be designated as *SSK* Part II, the *Sid. Sam* (Jagannātha's translation of the *Almagest*) being regarded as *SSK* Part I.

[12] We have corrected the printed *MC* Sanskrit text slightly

[13] R. C. Gupta, "The Hindu Method of Solving Quadratic Equations" *Jobit* (*Journal of the Birla Institute of Technology, Ranchi*), 1966-67, pp. 26–28.

[14] *Sam. Sid.*, Vol. II, pp. 1054–55.

[15] C. Ptolemy, *Almagest*, p. 14; English translation by R. C.Taliaferro, *Great Books of the Western World*, No. 16, 1952.

The Sanskrit rendering of Ptolemy's construction and demonstration occurs on pp. 16–17 of *Sam. Sid.*, Vol. I.