MĀDHAVA'S SINE AND COSINE SERIES

A. K. Bag

Indian National Science Academy
1 Park Street Calcutta 700 016

(*Received 29 April 1975*)

Indian scholar Mādhava (1350-1410) gave a table of almost accurate values of half-sine chords for twenty-four arcs drawn at equal intervals in a quarter of a given circle. The paper discusses the basis of arriving at such accurate values and has shown that Mādhava established the following sine and cosine series before Newton (1642-1727), De Moivre (1707-38) and Euler (1748), and used these relations to compute his table. These are :

$$\sin \theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots$$

$$\cos \theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \ldots$$

The rational of the relations has also been discussed in a nutshell in the paper.

The originality of Mādhava[1] (1350-1410) in astronomy, particularly in the application of refined mathematics in it in medieval period, is now being slowly recognised. He has given values of 24 half-sine chords for twenty-four arcs in a quarter of a circle drawn at equal intervals of 225′, viz. 225′, 450′, 675′, ... 5400′. The corresponding twenty-four sine values given by him are as follows.[2]

224′ 50″ 22‴	448′ 42″ 58‴	670′ 40″ 16‴
889′ 45″ 15‴	1105′ 1″ 39‴	1315′ 34″ 7‴
1520′ 28″ 35‴	1718′ 52″ 24‴	1909′ 54″ 35‴
2092′ 46″ 3‴	2266′ 39″ 50‴	2430′ 51″ 15‴
2584′ 38″ 6‴	2727′ 20″ 52‴	2858′ 22″ 55‴
2977′ 10″ 34‴	3083′ 13″ 17‴	3176′ 3″ 50‴
3255′ 18″ 22‴	3320′ 36″ 30‴	3371′ 41″ 29‴
3408′ 20″ 11‴	3430′ 23″ 11‴	3437′ 44″ 48‴

These values are correct to more or less eight to nine places of decimals. How Mādhava arrived at such accurate values of sine table, has been discussed in the paper.

The following passage found in the *Tantra-saṃgraha*[3] (1501 A.D.) has left distinct hints that the results contained in the lines were of Mādhava. The verses run as follows :

nihatya cāpa vargeṇa cāpam tattatphalāni ca/
haret samūlayugvargaistrijyāvargahataiḥ kramāt//
cāpam phalāni cādhodhonyasyoparyupari tyajet/
jīvāptyai, saṅgraho 'syaiva vidvān-ityādinakṛtaḥ//
nihatya cāpavargena rūpam tattatphalāni' ca/
hared vimulayugvargaistrijyāvargahataiḥ kramāt//
kintu vyāsadalenaiva dvighnenādyam vibhājyatām/
phalānyadhodhaḥ kramaśo nyasyoparyupari tyajet//
śarāptyai, saṅgraho 'syaiva stenastrītyādinā kṛtaḥ/

English Translation : Multiply the arc by the square of itself (multiplication being repeated any number of times) and divide the result by the product of the square of even numbers increased by that number and square of the radius (the multiplication being repeated same number of times). The arc and the results obtained from above are placed one below the other and are subtracted systematically one from its above. These together give the *jīvā* ($r \sin \theta$) collected here as found in the expression beginning with *vidvān* etc. Multiply the unit (i.e. radius) by the square of the arc (multiplication being repeated any number of times) and divide the result by the product of square of even number decreased by that number and square of the radius (multiplication being repeated same number of times). Place the results one below the other and subtract one from its above. These together give the *śara* ($r - r \cos \theta$), collected here as found in the expression beginning with *stena*.

If t_n and t'_n be the n-th expression for *jīvā* and *śara*, then for a small arc s, and radius r,

$$t_n = \frac{s^{2n} \cdot s}{(2^2+2)(4^2+4)\dots[(2n)^2+2n]r^{2n}} \qquad (n = 1, 2, 3, \dots).$$

The successive terms $t_1, t_2, t_3 \dots$ are,

$$t_1 = \frac{s^3}{3!\,r^2},\ t_2 = \frac{s^5}{5!r^4},\ t_3 = \frac{s^7}{7!r^6},\ t_4 = \frac{s^9}{9!r^8},\ \dots$$

Then according to the rule,

$$\begin{aligned} j\bar{\imath}v\bar{a} &= (s-t_1)+(t_2-t_3)+(t_4-t_5)+\dots \\ &= s - \frac{s^3}{3!r^2} + \frac{s^5}{5!\,r^4} - \frac{s^7}{7!\,r^6} + \frac{s^9}{9!\,r^8} - \frac{s^{11}}{11!r^{10}} + \dots \qquad \dots (1) \end{aligned}$$

Again, $$t_n' = \frac{s^{2n} \cdot r}{(2^2-2)(4^2-4)\dots[(2n^2)-2n]r^{2n}} \quad (n = 1, 2, 3, \dots)$$

The successive terms $t_1', t_2', t_3', \ldots$ are :

$$t_1' = \frac{s^2}{2!\, r}, \quad t_2' = \frac{s^4}{4!\, r^3}, \ldots \quad t_6' = \frac{s^{12}}{12!\, r^{11}} \ldots$$

As per rule, *śara* $= (r - t_1') + (t_2' - t_3') + \ldots \quad \ldots$

$$= r - \frac{s^2}{2!r} + \frac{s^4}{4!r^3} - \frac{s^6}{6!\, r^5} + \frac{s^8}{8!\, r^7} - \frac{s^{10}}{10!\, r^9} + \frac{s^{12}}{12!\, r^{11}} - \quad \ldots \quad (2)$$

when $s = r\,\theta$, the eqns (1) and (2) reduce to

$$\left.\begin{aligned} \sin\theta &= \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \frac{\theta^9}{9!} - \ldots \\ \cos\theta &= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \frac{\theta^8}{8!} - \ldots \end{aligned}\right\} \quad \ldots \quad (3)$$

Fortunately the passages beginning with *vidvān* and *stena* referred to in the above verses have been preserved in both *Āryabhaṭīyabhāṣya* of Nīlakaṇṭha (1443-1545) and *Karaṇapaddhati*.[5] In the former it has been clearly stated that the values of the first five terms t_5, t_4, t_3, t_2, t_1 of the eqn (1) and of $t_6', t_5', t_4', t_3', t_2'$, and t_1' of eqn (2) were given by Mādhava (*evāha mādhavaḥ*) when $s = 5400'$ and $r = 3437'\ 44''\ 48'''$. The values are : *vidvān* ($= 44''' = t_5$), *tunna bala* ($= 33''\ 6''' = t_4$), *kavīśanicaya* ($= 16'\ 5''\ 41''' = t_3$), *sarvārthaśīlasthiro* ($= 273'\ 57''\ 47''' = t_2$), *nirvirdhāṅganarendrarung* ($= 2220'\ 39''\ 40''' = t_1$) and *stena* ($= 6''' = t_6'$), *stripiśuna* ($= 5''\ 12''' = t_5'$), *sugandhinaganud* ($= 3'\ 9''\ 37''' = t_4'$), *bhadrāṅgabhavyāsana* ($= 71'\ 43''\ 24''' = t_3'$), *mīnāṅganarasiṃha* ($= 872'\ 3''\ 5''' = t_2'$), *unadhanakṛtbhūreva* ($= 4241'\ 9''\ 0''' = t_1'$).

These values when substituted in eqn (1) containing terms from t_1 to t_5, *jīvā* comes out to be $3437'\ 44''\ 48'''$, the 24th sine value given in the table of Mādhava (here $s = 5400'$). Similarly if s is replaced gradually by $225'$, $450'$, $675'$ ... Mādhava's sine table is obtained. Proceeding in a similar way and substituting values in eqn(2), the cosine table is obtained. This evidently shows that Mādhava, followed by the authors of *Tantrasaṃgraha* and *Karaṇapaddhati*, used the eqns (1) and (2) for the computation of the sine and cosine tables.

How Mādhava arrived at the equations (1) and (2) is not yet definitely known. The *Tantrasaṃgraha* (ch. 2, verse $12\frac{1}{2}$) of Nīlakaṇṭha and *Karaṇapaddhati* (ch. 6, verse 19) have both given that for small arc, *jīvā* $= s - \frac{s^3}{3!\, r^2}$ (approximately). The *Yuktibhāṣā*[6] has given the complete rational of the eqns (1) and (2). Its author Jyeṣṭhadeva(c. 1500-1600) in an effort to find an expression for the difference between any arc and its sine chord, divided the circumference of the quarter of a circle into

n equal divisions and considered the first and second sine differences. He then found the sum of the first n sine differences and cosine differences by considering all sine chords to be equal to corresponding arc and the small unit of the circumference to be equal to one unit, which evidently gives $jīvā = s - \frac{s^3}{3!r^2}$ and $śara = \frac{s^2 r}{2!r^2}$.

Since sine values are not actually equal to its arc length, further correction was applied *ad-infinitum* to each of the terms of the values obtained for *jīvā* and *śara*, which ultimately gives rise to the eqns (1) and (2). It would not be quite unlikely to presume that the rational was first established by Mādhava before Jyeṣṭhadeva could make use of it.

In Western mathematics Newton(1642-1727) is often given credit for the expansion of sine and cosine series No. (3). The result was established later algebraically on a solid foundation by De Moivre (1707-38) and Euler (1748)[7]. It is clear from the discussion that the Indian scholar Mādhava (1350-1410) used and possibly established the series (1), (2) and (3) of course in finite form before Newton, De Moivre and Euler, and laid the foundation of his sine table.

I express my gratitude to Sri S. N. Sen for his kind interest in the paper.

Notes and References

[1] Sarma, K. V. The date of Mādhava, a little known Indian astronomer, *Quarterly Journal of the Mythic Society*, **49**, 183-86, 1958.

[2] Vide *Tantrasaṃgraha*, edited by S. K. Pillai, Trivandrum Sanskrit Series 188, p. 19, Trivandrum 1958; vide also Rai, R. N. Sine values of the *Vaṭeśvarasiddhānta*, *Indian Journal of History of Science*, **7**, p. 11, 1972.

[3] Vide Yuktibhāṣā, Part I, edited with notes by Ramavarma (Maru) and Tampuran and A. R. Akhilesvare Iyer, p. 190, Trichur, 1948.

[4] Trivandrum Sanskrit Series No. 101 (*Gaṇitapāda*), p. 113; *Yuktibhāṣā*, p. 145.

[5] Ch. 6, verses 14-15 (for edition vide Trivandrum Sanskrit Series, No. 126); ch. 6, verse 7 refers to the circle of radius- $\frac{21600 \times 10^{10}}{31, 41, 59, 26, 536} = 3437' \, 44'' \, 48'''$, used for the computation of sine and co-sine table.

[6] Saraswathi, T. A., The Development of Mathematical Series in India, *Bulletin of the National Institute of Sciences of India*, **21**, pp. 338-42, 1963.

[7] Moritz Cantor, *Vorlesungen uber Geschichte der Mathematik*, Vol. 3, pp. 62-78, Leipzig, 1898 Smith, D. E. *A source Book in Mathematics*, Vol. 2, pp. 440-54, Dover Publication, 1959.