ON CERTAIN MATHEMATICAL TOPICS OF THE *DHAVALĀ* TEXTS

(The Jaina School of Mathematics)

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This article introduces in brief certain mathematical techniques adopted in manipulation of finite and transfinite sets, appearing in the third volume of Vīrasenācārya's *Dhavalā* commentary (ninth century A.D.) of *Ṣaṭkhaṇḍāgama* which was composed by Puṣpadanta and Bhūtabali (c. second century A.D.). Various sets have been operated upon by logarithmic functions to the base two, by squaring and extracting square roots, and by raising the powers to finite or infinite. The operations have been repeated over and over to yield new sets, some times infinite in appropriate cases. The concepts of measures are then elaborated through various relations. Here description is mainly about a set of souls, transfinite in number and having mythic vision. First the measure of this set is illustrated relative to fluent, time, quarter, and becoming. Then the set has been analysed through the methods of cut, division, reduction, measure, reason, explanation, and abstraction. The lower abstraction is in the forms of the dyadic sequences of square,cube and cube-non-cube types. The upper abstration is also of the types of dyadic sequences which are denominated as square, cube, cube-non-cube, ordinarily and in adopted, adopted-adopted as well as adopted-multiplier ways.

1. Introduction

The *Dhavalā* commentary of Vīrasenācārya (ninth century A.D.) contains many valuable information on mathematics. It is proposed to supplement the two articles, "Mathematics of the *Dhavalā*" by Singh[2], and to suggest clarification of certain operations, not attempted by him.

The Jaina source work, *Ṣaṭkhaṇḍāgama*, on which Vīrasena supplied his commentary, was written in the second century A.D. The school of Vardhamāna Mahāvīra, similar to the school of Pythagoras, noted for its mathematico-philosophic pursuits, flourished in India, closely round about the same period[3]. The results presented here are all based on geometrical concepts prevalent in the school.

The epistemological concepts were founded on the theories of fluents (*dravyas*) and tautos (*tattvas*[4]). The fluents were defined to have certain common controls (*guṇas*) and distinguishing controls as well. The numbers were also ascribed to these fluents, their controls and their events (*paryāyas*). Then a type of quantum formalism and a mathematical system theory was developed.

We shall present here only the rudiments about soul (*jīva*), matter (*pudgala*) motion and rest. Ultimate particle of matter, a fluent, carries with it a certain set of positive or negative indivisible-corresponding-sections (*avibhāgī praticcheda*) of charge-levels (*aṃśas*) of touch (*sparśa*) control.

The three-dimensional E^3-space (*ākāśa*) is divided into empty and non-empty regions. The portion containing all the fluents is called universe (*loka*), and the empty portion is called non-universe (*aloka*)[5]. The space occupied by the ultimate particle is called a point (*pradeśa*), regarded as indivisible. On the basis of this metric various topological spaces could form a subject of study in this school.[6]

The motion of souls and matter particles is cooperated by aether (*dharma*), a fluent which may be called a motion-casuality-continuum, existing coincident with the E^3-space. This makes a boundary of the universe beyond which there cannot be any motion[7]. The state of rest is cooperated by another similarly coincident fluent, called anti-aether (*adharma*) which may be called rest-causality-continuum.

The modification-causality fluents are as many as there are points in the universe, with which they are coincident, filling up the non-empty E^3-space continuum. This is thus a discrete granular structure of time-atoms (*kālāṇus*). Thus each time-atom coincides with a E^3-space point. Such a structure defines a line known as a world-line (*jagaśreṇī*). The existence of such world-lines throughout the E^3-space defines a Cartesian frame on the basis of motions of souls[8].

It is important to see how the "now of Zeno" or "the present" or an instant (*samaya*) is defined as an indivisible interval of time[9]. It may also be called an ultimate part of relative-time (*vyvahāra kāla*). The Instant is defined as the time taken by an ultimate-particle to traverse its way from a point to the next with a minimum velocity, During the same instant it traverses at set of more points with a greater velocity along a world line, it being omnipresent at every one of the points.

There is one more relation between the set ϕ of points in a fiinger (*aṅgula*) width and the set T_h of instants in a hollow (*palya*)-measure of time :

$$\log_2(\phi) = (\log_2(T_h))^2 \qquad \ldots \quad (I)^{10}$$

Thus the set of points and instants filling up finite and infinite gaps of space and time continuums stand as definite means of measures, introducing topology.

The cosmological unit of measure of distance is rope (*rajju*), reminding us of the Egyptian rope-stretchers or Harpedonaptae[11]. Virasenācārya was confronted with a controversy regarding the measure of volume of universe as well as that of *rajju* in terms of the number of E^3-space points[12]. He established the following result on the basis of the method of mediatio and logarithmic computation :

$$R = 2^{(n+s+\log \xi)} \qquad \ldots \quad (\text{I.I})^{13}$$

where R is the cardinal number of points contained in a *rajju*, n is the number of circular islands and oceans in shape of rings, s is a certain summable (*saṃkhyeya*) number, and ξ is the cardinal number of points contained in the width of Jambūdvīpa. [The right hand side is two raised to the power $n+s+\log \xi$].

Regarding the use of logarithms to any base, Singh is first to recognize *Arddhaccheda* as logarithm to the base two[14]. In the above, logarithm to the base 2 of a finite segment is the number of points (bisection-cuts) at which the segment is continually bisected sticking to any one of the sides always. While finding out the $\log_2$ of a term lying between 2^{n-1} and 2^n, they took the value either as $n-1$or n and an error of 1 was considered negligible for the non-summable n. In the physical nature of things, they did not think it reasonable to divide the postulated indivisible E^3-space point[15]. That is why the school dealt with many infinitness on the field of an integral domain and range extending from 1 to the biggest set of indivisible-corresponding-sections of omniscience (*Kevala Jñāna*). In order to denote these integers the word *salāgā* was used, which means a log or a small piece of wood or other material needed for counting. The question as to why Napier called *arddhoccheda* or *triakccheda salāgās* as logarithm to the base two or three may now be answerd better after a study of corresponding topological sequences (*dhārās*) in the school[16].

Apart from *rajju*, one may find various other similar historically correlating factors in Indo-Egyptian mathematical studies through Prakrit texts. A few may be mentioned for example : the method of mediatio and duplatio; hand measure; "*aha*"[17] calculations resembling calculus of sets (*rāśi gaṇana*);[18] "*sha*"[19] possibly a circular cylinder, resembling *salāgā kuṇḍa*"; and the values of π.

Besides the above set theoretic approach in geometry, the number measure[20] also is worthy of mention as it is based on ordered structures of numbers and on union of sets as well as process of *Vargita-Saṃvargita*[21]. When a set or a quantity is multiplied to itself as many times as itself, it is said to be *Vargita-saṃvargita* one time.

The details of the procedure for obtaining various types of number measure sets may be available in various texts[22], particularly in the article by Singh. He developed his own symbolism for this purpose. He expressed *ananta* (infinite) by A, and *asaṃkhyeya* (non-summable) by a[23]. However, it was felt more convenient to replace them by $\mathscr{J}$ and $\mathscr{A}$[24].

Herein and what follows, the following symbols and notations will be used.

Notation	Meaning
$\mathcal{I}_{ij}$	juniormost infinite-infinite (*jaghanya anantānanta*)
$\overline{\mathcal{I}_{ij}}\rvert^{1}$	$(\mathcal{I}_{ij})^{\mathcal{I}_{ij}}$
$\overline{\mathcal{I}_{ij}}\rvert^{2}$	$\left\{(\mathcal{I}_{ij})^{\mathcal{I}_{ij}}\right\}^{\left\{(\mathcal{I}_{ij})^{\mathcal{I}_{ij}}\right\}}$
$\overline{\mathcal{I}_{ij}}\rvert^{3}$	$\overline{\left\{(\mathcal{I}_{ij})^{\mathcal{I}_{ij}}\right\}^{\left\{(\mathcal{I}_{ij})^{\mathcal{I}_{ij}}\right\}}}\Bigg\rvert^{1}$
$\mathcal{I}_{pj}$	juniormost perdu-infinite (*jaghanya parītānanta*)
$\mathcal{A}$	non-summable (*asaṃkhyeya*)
$\mathcal{A}au$	uppermost non-summable-non-summable (*utkṛṣṭa asaṃkhyeyāsaṃkhyeya*)
log	logarithm
$\mathcal{I}im$	medium infinite-infinite (*madhyama anantānanta*)
$\mathcal{I}iu$	uppermost infinite-infinite (*utkṛṣṭa anantānanta*)
$\mathcal{I}$	infinite (*ananta*)
ζ	finite or summable (*saṃkhyeya*)

2. Logarithmic Treatment

In Prakrit texts the operations among the sets are worth study. The sets treated therein are finite, transfinite, ordered, well-ordered, plain and mixed. We shall call a set mixed when it has been formed as a result of mixing well-ordered set or sets with plain or ordered set or sets. In a plain set the order of terms is ignored, the problem arising here is whether it is justified to go along with an ordinal of a mixed set[25]? Cantor attempted to get rid of this problem by establishing the well-ordering theorem[26]. The Jaina School of Mathematics indirectly made use of such a theorem in setting forth topological sequences locating most of the finite and transfinite sets.[27]

In what follows, Vīrasenācārya has attempted to prove that $\overline{\mathcal{J}_{ij}}|^3$ is infinite times less than the cardinal L of the set of all living beings in the whole universe[28]. We are making use of the word "cardinal" in the sense that may carry an analogical significance of the modern usage. Vīrasenācārya proceeds as follows :

$$\log \mathcal{J}_{pj}+1+\log(\log \mathcal{J}_{pj}) = \log(\log \mathcal{J}_{ij}) \quad \ldots \quad (2.01)$$

Herein and what follows, the base of logarithm is two.

$$(2 \log \mathcal{J}_{pj})(\mathcal{J}_{pj}) = \log \mathcal{J}_{ij}, \quad \ldots \quad (2.02)$$

because,

$$\mathcal{J}_{ij} = [\mathcal{J}_{yj}]^2 = [(\mathcal{J}_{pj})\ \mathcal{J}_{pj}]^2.$$

Further,

$$\log \mathcal{J}_{ij} = \mathrm{A}(\mathcal{J}_{pj}) \quad \ldots \quad (2.03)$$

or

$$\log \ \mathcal{J}_{ij} = [\mathcal{J}_{pj}{}^2] \div \mathcal{A} \quad \ldots \quad (2.04)$$

where $\mathcal{A}$ is non-summable, because

$$\log \ \mathcal{J}_{ij} = (2 \log \mathcal{J}_{pj})(\mathcal{J}_{pj})$$

and it is known that $\mathcal{J}_{pj} = \mathcal{A}\mathrm{au}+1$.

Further,

$$\log \mathcal{J}_{ij} = (2 \log \mathcal{J}_{pj})(\mathcal{J}_{pj}) = \mathcal{J}_{pj} \log(\mathcal{J}_{pj})^2$$

$$= (\mathcal{J}_{pj})^2 \frac{2 \log(\mathcal{J}_{pj})}{\mathcal{J}_{pj}}$$

and find that Vīrasena might have recognized

$$\frac{2 \log(\mathcal{J}_{pj})}{\mathcal{J}_{pj}} \text{ or } \frac{2 \log(\mathcal{A}au+1)}{\mathcal{A}au+1} \text{ as } \frac{1}{asaṃkhyāta}.$$

Now it is known that

$$\log \mathcal{J}_{ij} > \log \mathcal{J}_{pj}\ , \quad \ldots \quad (2.05)$$

and

$$\log \mathcal{J}_{ij}+\log(\log \mathcal{J}_{ij}) = \log(\log \overline{\mathcal{J}_{ij}}|^1), \quad \ldots \quad (2.06)$$

$$\mathcal{J}_{ij}(\log \mathcal{J}_{ij}) = \log \overline{\mathcal{J}_{ij}}|^1. \quad \ldots \quad (2.07)$$

Therefore,

$$\log \overline{\mathcal{J}_{ij}}|^1 = \mathcal{J}[\mathcal{J}_{ij}] \quad \ldots \quad (2.08)$$

because log $\mathcal{J}_{ij}$ is some transfinite number.

Similarly,

$$\log \overline{\mathcal{J}_{ij}}|^{1} = \frac{[\mathcal{J}_{ij}]^2}{\mathcal{J}} \qquad \ldots \quad (2.09)$$

Further,

$$\log \overline{\mathcal{J}_{ij}}|^{1} + \log\log \overline{\mathcal{J}_{ij}}|^{1} = \log\log \overline{\mathcal{J}_{ij}}|^{2} \qquad \ldots \quad (2.10)$$

Also.

$$(\log \overline{\mathcal{J}_{ij}}|^{1})(\overline{\mathcal{J}_{ij}}|^{1}) = \log \overline{\mathcal{J}_{ij}}|^{2}, \qquad \ldots \quad (2.11)$$

therefore,

$$\log \overline{\mathcal{J}_{ij}}|^{2} = \mathcal{J}(\overline{\mathcal{J}_{ij}}|^{1}), \qquad \ldots \quad (2.12)$$

where $\mathcal{J}$ is some transfinite number.

Similrly

$$\log \overline{\mathcal{J}_{ij}}|^{2} = \frac{[\overline{\mathcal{J}_{ij}}|^{1}]^2}{\mathcal{J}}. \qquad \ldots \quad (2.13)$$

It is known that,

$$\log \overline{\mathcal{J}_{ij}}|^{2} + \log\log \overline{\mathcal{J}_{ij}}|^{2} = \log\log \overline{\mathcal{J}_{ij}}|^{3} \qquad \ldots \quad (2.14)$$

Now,

$$\log(\log \overline{\mathcal{J}_{ij}}|^{3}) < [\overline{\mathcal{J}_{ij}}|^{1}]^2.$$

because according to (2.11) and (2.14),

$$\log\log \overline{\mathcal{J}_{ij}}|^{3} = [\mathcal{J}_{ij}]^{\mathcal{J}_{ij}+1} \log \mathcal{J}_{ij} + [\mathcal{J}_{ij}+1] \log \mathcal{J}_{ij} + \log\log \mathcal{J}_{ij}$$

Thus the cardinal $\log\log \overline{\mathcal{J}_{ij}}|^{3}$ has not reached even a single square-place (*varga-sthāna*) above $\overline{\mathcal{J}_{ij}}|^{1}$. From this, the author concludes that

$$\log\log [\overline{\mathcal{J}_{ij}}|^{1}]^2 = \log\log\log\log \overline{\mathcal{J}_{ij}}|^{3}. \qquad \ldots \quad (2.16)$$

The left hand side of (2.16) is also stated as,

$$\begin{aligned} \log\log [\overline{\mathcal{J}_{ij}}|^{1}]^2 &= \log(2 \log \overline{\mathcal{J}_{ij}}|^{1}] \\ &= 1 + \mathcal{A}' \mathcal{J}_{pj} + \log \mathcal{J}_{pj} + 1 + \log\log \mathcal{J}_{pj} \\ &= \mathcal{A}\, \mathcal{J}_{pj}. \end{aligned} \qquad \ldots \quad (2.17)$$

Now Virasena applies the method of reductio ad absurdum :

If we take it for granted that

$$\log\log \overline{\mathcal{J}_{ij}}|^{3} = \log\log (L),$$

in which case,

$$\overline{\mathcal{J}_{ij}}|^{3} = L$$

which is not so, because according to *parikarma*,

$$\log\log (L) = [\{(\mathcal{J}_{ij})^2\}^2]^{2\cdots}, \qquad \ldots \quad (2.18)$$

the indices raised $\mathscr{J}_{im}$ times, whereas,

$$\log \log \overline{\mathscr{J}_{ij}}\,|^{3} \neq [\{(\mathscr{J}_{ij})^{2}\}^{2}]^{2\cdots}, \qquad \ldots (2.19)$$

the indices being raised $\mathscr{J}_{im}$ times.

As a matter of fact,

$$\log \log \mathscr{J}_{ij}\,|^{3} = (\mathscr{J}_{ij})^{2(\mathscr{A}\mathscr{J}_{pj})} \qquad \ldots (2.20)$$

Vīrasena now proceeds to prove (2.20) as follows :

It is known that $\log \log \mathscr{J}_{ij} > \log \mathscr{J}_{pj}$, ... (2.21)

because, $\mathscr{J}_{ij} = [(\mathscr{J}_{ij})\ \mathscr{J}^{ij}]^{2}$,

therefore $\log \mathscr{J}_{ij} = (2 \log \mathscr{J}_{pj})\ \mathscr{J}_{pj}$,

therefore $\log \mathscr{J}_{ij} > \mathscr{J}_{pj}$

and hence the (2.21).

Again,

$$\log \log \overline{\mathscr{J}_{ij}}\,|^{3} < [\overline{\mathscr{J}_{ij}}\,|^{1}]^{2}$$

by virtue of the relation (2.15).

Therefore,

$$\log \log \log \log \overline{\mathscr{J}_{ij}}\,|^{3} < \log 2\mathscr{J}_{ij} + \log \log \mathscr{J}_{ij}$$
$$\ll [\mathscr{J}_{ij}]^{2}. \qquad \ldots (2.22)$$

Now,

$$\log \log \log \log \overline{\mathscr{J}_{ij}}\,|^{3} < 1 + \log \mathscr{J}_{ij} + \log \log \mathscr{J}_{ij},$$

therefore $\log \log \log \log \log \log \overline{\mathscr{J}_{ij}}\,|^{3} - \log \log \mathscr{J}_{ij} < 1 + 2\,\mathscr{J}_{pj} \log \mathscr{J}_{pj}$. ... (2.22a)

At the same time, we have

$$\log \log \overline{\mathscr{J}_{ij}}\,|^{3} > \overline{\mathscr{J}_{ij}}\,|^{1} \text{ by virtue of (2.15).}$$

Therefore,

$$\log \log \log \log \overline{\mathscr{J}_{ij}}\,|^{3} > \log \mathscr{J}_{ij} + \log \log \mathscr{J}_{ij},$$

or

$$\log \log \log \log \overline{\mathscr{J}_{ij}}\,|^{3} - \log \mathscr{J}_{ij} > \log \log \mathscr{J}_{ij}$$
$$> 2\mathscr{J}_{pj} \log \mathscr{J}_{pj}. \qquad \ldots (2.22b)$$

Now if in (2.22a) and (2.22b), $\mathscr{A}$ is substituted in place of 2 log $\mathscr{J}_{pj}$, then

$$\log \log \log \log \overline{\mathscr{J}_{ij}}\,|^{3} - \log \log \mathscr{J}_{ij} = \mathscr{A}.\ \mathscr{J}_{pj}, \qquad \ldots (2.23)$$

where $\mathscr{A}$ is *asaṃkhyāta* (nonsummable).

Again, from (2.20),

$$\log\log\log\log \overline{\mathcal{J}_{ij}}\,]^{3} = \log(2^{\mathcal{A}(\mathcal{J}_{pj})}\ \log \mathcal{J}_{ij})$$

$$= \mathcal{A}\mathcal{J}_{pj}\log 2 + \log\log \mathcal{J}_{ij},$$

therefore $\log\log\log\log \overline{\mathcal{J}_{ij}}\,]^{3} - \log\log \mathcal{J}_{ij} = \mathcal{A}.\mathcal{J}_{pj}$ ... (2.24)

which is the same as (2.23).

Thus the proof is evident from the relations (2.18) and (2.20)

In the above treatment, it is evident that the logarithmic treatment of the finite cardinals (although non-summable in measure) has been effected. $\mathcal{J}_{ij}$ has been formally denominated as infinite though actually it is not so. The only transfinite cardinal of which logarithm to the base 2 has been considered is L.

3. Measure and Analytical Methods

In an appaoach to substance measure (*dravya-pramāṇānugama*), the measure has been described relative to fluent, time, quarter, and becoming[29]. The present treatment has been limited to the exposition of the measure of the set of souls who have mythic vision (*mithyā dṛṣṭi jīva rāśi*).

Relative to fluent-measure, the souls having mythic vision, are infinite.[30] The measure of this infinite has been shown to be $\mathcal{J}_{im}$ by Vīrasena, where this cardinal number is said to lie between

$$[\{(\mathcal{J}_{ij})^2\}^2]^{2\cdots} < \mathcal{J}_{im} < [\{(\mathcal{J}_{iu})^{\frac{1}{2}}\}^{\frac{1}{2}}]^{\frac{1}{2}\cdots} \qquad \ldots\ (3.01)$$

where the process of squaring is ad infinitum[31].

Here the word "fluent" designates either "soul" or "matter"[32]. In general there are two types of fluents, soul and non-soul. This type of measure appears to be rather a number-measure which has been discussed in details[33]. In what follows we shall denote this set of souls by L_{c1}.

It may be noted that in the present treatment, a set and its cardinal will not be distinguished as had been the style in the Jaina School of Mathematics.

Relative to time, L_{c1} can not be exhausted by the set of instants (*samaya rāśi*) of the past. We shall denote it by T^-.[34] This set of instants is ordered and has $^*\omega$ for its order type. Its elements are given as (..., 3, 2, 1). There is no first element.

For comparing the measures of the above two sets, the method of one to one correspondence has been used.[35]

It is shown that

$$T^- < L_{c1}, \qquad \ldots\ (3.02)$$

and in order to remove any doubt he relates the results of the mixed type of comparability about sixteen sets.[36]

The present is a singlet set containing only one instant of time. It has been been regarded as the smallest set though there are concepts of null sets. Denoting it by T^0, we have,

$$L_e^- = \mathcal{J}_{yj}(T^0) \quad \ldots \quad (3.03)$$

where L_e^- is the set of the souls incapable of being accomplished ever due to their tendency known as "*abhavyatva*".

Further,

$$T_{L_e 0} = L_{e^-} \frac{(T^- \div \mathcal{J})}{\left[1 \frac{6}{8} T_m^* \right]} \quad \ldots \quad (3.04)$$

where $T_{L_e 0}$ is a set of instants, known as the accomplishing period (*siddha-kāla*), and T_m^* is the set of instants contained in a month.

A doubt regarding the establishment of the measure of $T^- \div \mathcal{J}$ has been refuted by Vīrasena here. He argues, "In case its measure is not established, the question of its non-existence will arise, which is not so. Again if we are able to know its beginning, it will be called, "having beginning" and not ab aeterno."[37]

Further,

$$L_{e0} = \zeta(T_{L_e 0}) \quad \ldots \quad (3.05)$$

where L_{e0} represents the set of the accomplished souls.

$$\zeta(T_r)(L_{e0}) = T_{L_e \theta} \quad \ldots \quad (3.06)$$

where T_r denote the set of instants contained in a single trail (*āvalikā*), and $T_{L_e \theta}$ is non-accomplishing period.

$$T_{L_e 0} + T_{L_e \theta} = T^- \quad \ldots \quad (3.07)$$

Now,

$$T^- . \frac{L_{c_1 e+}}{\mathcal{J}} = L_{c_1 e^+} \quad \ldots \quad (3.08)$$

where Lc_1e^+ is the set of souls who, inspite of having mythic vision at present, will be accomplished in future (*bhavya mithyā dṛṣṭi jīva rāśi*)

Then,

$$Lc_1e^+ + \sum_{n=2}^{n=14} Lc_n = L_e^+ \quad \ldots \quad (3.09)$$

where L_e^+ is the set of souls who are accomplishable, and $\sum_{n=2}^{n=14} Lc_n$ is the set of the souls in the remaining thirteen control stations;

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whence,

$$(L_e{}^{+} - \sum_{n=2}^{n=14} Lc_n) + Le^{-} = Lc_1. \qquad \dots \quad (3.10)$$

Further,

$$L - L_e{}^{0} = Lc_1 + \sum_{n=2}^{n=14} Lc_n. \qquad \dots \quad (3.11)$$

Here $L-L_e{}^0$ is the set of all the worldly souls. We may also represent it symbolically as L_w.

Thus,

$$L_w = L - L_e{}^0 \qquad \dots \quad (3.12)$$

Now,

$$L \,.\, \mathcal{J} = M, \qquad \dots \quad (3.13)$$

where M is the set of ultimate particles of matter contained in the universe (*loka*).

Then,

$$M . \mathcal{J} = T^{+} \qquad \dots \quad (3.14)$$

where T^{+} is the set of instants in the future.

Henceforth,

$$T^{-} + T^{0} + T^{+} = T, \qquad \dots \quad (3\text{'}15)$$

where T is the set of instants contained in whole time. It may be noted that T^{+} has for its order type ω, and may be represented as, $(1, 2, 3, \dots)$, which shows that it has no last element.

Further,

$$T. \mathcal{J} = (\sigma^0)^3 \qquad \dots \quad (3.16)$$

where $(\sigma^0)^3$ is the set of space-points contained in the non-universe or empty space (*alokākāśa*), and

$$(\sigma^0)^3 + \{(\sigma)^3 - (\sigma^0)^3\} = (\sigma^0)^3 + (\lambda)^3 = (\sigma)^3 \qquad \dots \quad (3.17)$$

where λ^3 is the set of space-points contained in the universe (*lokākāśa*) and $(\sigma)^3$ is the set of space points contained in whole space (*ākāśa*).

From (3.07) we have, by virtue of (3.06) etc.,

$$\begin{aligned} T^{-} &= T_{L_e{}^e} + T_{L_e{}^0} \\ &= \zeta T_r\, L_e{}^0 + T_{L_e{}^0} \\ &= (\zeta\zeta T_r + 1) T_{L_e{}^0} \end{aligned} \qquad \dots \quad (3.18)$$

Now, from (3.10), we have,

$$L_{c_1} = L_{e^+} - \sum_{n=2}^{14} L_{c_n} + L_{e^-} = L_{c_1 e^+} + L_{e^-} \quad \text{by virtue of (3.09),}$$

$$= T^{-} \frac{L_{c_1 e^+}}{\mathcal{J}} + \mathcal{J}_{yj}(T^0) \quad \text{(by virtue of (3.08) and (3.03),}$$

$$= T^{-} \frac{L_{c_1} - L_{e^-}}{\mathcal{J}} + \mathcal{J}_{yj}(T^0) \quad \text{(by virtue of (3.10),}$$

$$= T^{-} \left[\frac{L_{c_1}}{\mathcal{J}} \right] - \frac{\left(1 \frac{6}{8} m^* \right)}{1} . \, T_{Le^0} + \mathcal{J}_{yj}(T^0) \qquad \ldots \quad (3.19)$$

It appears from the text that $\mathcal{J}$ in the last expression does not stand for $\mathcal{J}_{tm}$.[38] We are unable to proceed ahead of this step, except that,

$$L_{c_1} = T^{-} \frac{L_{c_1}}{\mathcal{J}} - \frac{\left(1 \frac{6}{8} m^* \right) T^{-}}{\zeta \, \zeta \, T_r + 1} + \mathcal{J}_{yj} \, (T^0), \qquad \ldots \quad (3.20)$$

by virtue of (3.18).

However in all cases, $\mathcal{J}$ must stand for an infinite less than $\mathcal{J}_{tm}$, in which case, it is evident that $L_{c_1} > T^-$. The result shows that the L_{c_1} is not exhausted in spite of the fact that relative to the accomplished souls, L_w is continually decreasing.

Relative to quarter, the cardinal of the set L_{c_1} is said to be $\mathcal{J}_l$ times the cardinal of the set λ^3.[39]

Vīrasena illustrates this comparability through the method of mapping of L_{c_1} onto λ^3, i.e. by alloting to every space-point of the universe (λ^3) an element of L_{c_1}, and repeating the process $\mathcal{J}_l$ times[40]. The universe is 343 cubic ropes (*rajjus*). A *rajju* is a cosmological measure of the universe.. This length in a Euclidean flat space may be considered to be a straight line and the set of space-points contained in it may be denoted by $\frac{\lambda}{7}$ or by ρ.

We now proceed to illustrate the measure of ρ in terms of space-points, discussed by Vīrasena[41]. Let the integral number of islands and oceans be n and the diameter of Jambūdvīpa be denoted in terms of the set of space-points contained in the stretch, ξ. In the discussion the term, *guṇide* should be replaced by *bhaṇide*, otherwise, results obtained would be incorrect.

Thus, according to one of the schools,

$$2^{\{n+1+\log \xi\}} = \rho, \qquad \ldots \quad (3.21)$$

or by taking $\log_2$ both the sides,

$$(n+1+\log_2 \xi) \log_2 2 = \log_2 \rho \quad \dots \quad (3.22)$$

or

$$n+1+\log_2 \xi = \log_2 \rho. \quad \dots \quad (3.23)$$

If we follow "*guṇide*", we get the wrong results,

$$[2^{\{n+1+\log_2 \xi\}}]\{n+1+\log_2 \xi\} = \frac{\lambda}{7} = \rho, \quad \dots \quad (3.24)$$

because $\rho \log_2 \rho \neq \rho$.

Now according to the other school,

$$2^{\{n+\zeta+\log_2 \xi\}} = \rho. \quad \dots \quad (3.25)$$

It may be noted that in place of 1 in the power of 2 in (3.21), ζ has been taken in (3.25).

If one insists on having *guṇide*, $\log_2 2$ will have to be interpreted for "*chiṇṇāvisiṭṭhama*" and therefore

$$\rho = 2^{\{n+\log \zeta+\log \xi\}} \log_2 2. \quad \dots \quad (3.26)$$

As every ocean has an even number as its order, every island would have an odd number for its label. Thus the *Svayambhū-ramaṇa* ocean would have $n = 2m$ for its label. Its corresponding island would have $n-1$ or $2m-1$ for its label. Now the diameter of the ocean is 2^{2m-1} lacs of *yojanas*, the measure of *rāju* in lacs of *yojanas* would be

$$1+2(2+2^2+2^3+\dots+2^{2m-1}). \quad \dots \quad (3.27)$$

The sum of the above series is

$$= 2^{2m+1}-3 \quad \dots \quad (3.28)$$

This is the measure according to the first school. Now we proceed on to find the value of $\log_2 \rho$ from the above, remembering that $\log_2$ of a lac of *yojana* will have to be determined ultimately in terms of space-points.

It is evident that the first middle point or section (*arddhaccheda*) of the *rāju*, the width of the miduniverse, would lie on the centre of the Jambūdipa, from where the distance of the outskirts of the *Svayambhūramaṇa* would be,

$$\tfrac{1}{2}+[2+2^2+2^3+\dots+2^{2m-1}]$$

$$= 2^{2m}-\frac{3}{2} \text{ lacs of } yojanas \quad \dots \quad (3.29)$$

The middle point or the mid-section of this distance given by (3.29) would lie on or outside of the corresponding island from the centre of the *Jambūdvīpa* at a distance of

$$\tfrac{1}{2}+[2+2^2+2^3+\ldots+2^{2m-2}] \text{ lacs of } yojanas, \text{ or, } 2^{2m-1}-\frac{3}{2} \text{ lacs of } yojanas. \quad \ldots \quad (3.30)$$

Dividing (3.29) by 2 we have $2^{2m-1}-\frac{3}{4}$ lacs of *yojanas*, ... (3.31)

and as

$$2^{2m-1}-\frac{3}{4} > 2^{2m-1}-3/2, \quad \ldots \quad (3.32)$$

therefore, the second midsection of the *rāju* will fall on the *Svayambhū ramaṇa* ocean. Similarly the third midsection or *arddhaccheda* will lie on the corresponding island, because the distance of the centre of ocean preceding the *Svayambhūramaṇa* island from its own outskirts is

$$\tfrac{1}{2}+[2+2^2+2^3+\ldots+2^{2m-3}]$$

$$= 2^{2m-2}-3/2 \quad \text{lacs of } yojanas, \quad \ldots \quad (3.33)$$

whereas the half of the amount given in (3.31) is

$$2^{2m-2}-\frac{3}{2^3} \quad \ldots \quad (3.34)$$

and,

$$2^{2m-2}-\frac{3}{2^3} > 2^{2m-2}-\frac{3}{2} \quad \ldots \quad (3.35)$$

Similarly it is obvious that,

$$2^{2m-3}-\frac{3}{2^4} > 2^{2m-3}-\frac{3}{2},$$

and in general,

$$2^{2m-(x-1)}-\frac{3}{2^x} > 2^{2m-(x-1)}-\frac{3}{2}, \quad \ldots \quad (3.36)$$

where x is the number of cuts or sections.

If we start from the *Svayambhūramaṇa* ocean, then the order of the "*Lavaṇa*" ocean would be $(2m-1)$. As the first cut lies on the centre of the Jambūdvipa the value of x is $2m$. Substituting this value of x in (3.36), we have,

$$2^{2m-(2m-1)}-\frac{3}{2^{2m}} > 2^{2m-(2m-1)}-\frac{3}{2}$$

or

$$2^1-\frac{3}{2^{2m}} > 2^1-\frac{3}{2} \quad \ldots \quad (3.37)$$

The result shows that the $2m$-th cut lies on the *Lavaṇa* ocean. This cut or section lies at a distance of $\frac{1}{2}+\frac{3}{2^{2m}}$ lacs of *yojanas* inside the *Lavaṇa* ocean from its outskirts It may be noted that n or 2 is some *asaṃkhyāta* number, therefore, in the limit, $\frac{3}{2^{(2m+1)}}$ may be ignored.

Thus after getting $2m+1$ cuts of the *rāju*, $1-\frac{3}{2^{(2m+1)}}$lacs of *yojanas* of distance is left. In the special commentary of the editors, it has been stated that after getting 17 cuts of a lac of *yojana*, a *yojana* is left. The statement is incorrect because

$$(2)^{17} = 131072 \qquad \ldots \quad (3.38)$$

The following statement is also not clear. In case after 19 cuts of a *yojana*, a *sūci aṅgula* is left, then there should be $(2)^{19}$ *sūci aṅgula* to be *pramāṇāṅgula*, still there are 768000 *pramāṇāṅgula* or 384000000 *utsedhāṅgula* in a *yojana*,

whereas,

$$(2)^{19} = 524288. \qquad \ldots \quad (3.39)$$

Further,

$$[T_h]^{\{\log_2 T_h\}} = \phi \qquad \ldots \quad (3.40)$$

or,

$$[\log_2 T_h]^2 = \log_2 \phi. \qquad \ldots \quad (3.41)$$

Thus the statement given in the special commentry, is not correct so far as the numbers 17 and 19 are concerned,

$$\{2m+1+17+19+(\log_2 T_h)^2\} = \log_2 \rho. \qquad \ldots \quad (3.42)$$

As per definition of a world-line (*jaga-śreṇi*), given on page 37 of the introduction, we have,

$$[(\phi)^3]^{\{T_h \div A\}} = \lambda = 7\rho, \qquad \ldots \quad (3.43)$$

where λ is the set of space-points contained in a linear stretch of length 7ρ.

According to the other school,

$$[(\phi)^3]^{\left\{\frac{\log T_h}{A}\right\}} = \lambda = 7\rho. \qquad \ldots \quad (3.44)$$

It is obvious that in case of identity, A in both the relations should be different. A space-point is defined on the basis of the space occupied by an indivisible part of matter. Thus $\log_2$ of a set of number of space-points lying between 2^{n-1} and 2^n was roughly taken to be either $n-1$ or n.

The fourth kind of measure is the becoming measure[42]. The knowledge of the three foregoing measures is the becoming measure. Vīrasena has described the same independently, perhaps on the basis of the traditional knowledge. The methods discussed by him are those of cut (*khaṇḍita*), division (*bhājita*), spread (*viralita*), reduction (*apahṛta*), measure (*pramāṇa*), reason (*kāraṇa*), explanation (*nirukti*), and abstration (*vikalpa*).[43]

The following description is that of the square-station (*varga sthāna*)[44]. First of all a polar set (*dhruva rāśi*) is established as follows :

The polar set

$$= L + L_{e^0} + \sum_{n=2}^{14} L_{c_n} + \frac{\left(L_{e^0} + \sum_{n=2}^{14} L_{c_n}\right)^2}{L_{c_1}}$$

$$= L_{c_1} + 2\left(L_{e^0} + \sum_{n=2}^{14} L_{c_n}\right) + \frac{\left(L_{e^0} + \sum_{n=1}^{14} L_{c_n}\right)^2}{L_{c_1}}$$

$$= \frac{\left(L_{c_1} + L_{e^0} + \sum_{n=2}^{14} L_{c_n}\right)}{L_{c_1}} = \frac{(L)^2}{L_{c_1}} \qquad \ldots \quad (3.45)$$

Vīrasena has given an arithmetical symbolic representation :

$$3 + \frac{9}{13} + 16 = \frac{(16)^2}{13} = \text{polar set}, \qquad \ldots \quad (3.46)$$

where $L_{c_1} = 13,\ L = 16$ and $L_{e^0} + \sum_{n=2}^{14} L_{c_n} = 3$

The method of cut[45]

If $(L)^2$ is cut so as to have $\frac{(L)^2}{L_{c_1}}$ divisions, then each such division is found to be L_{c_1}.

The method of division :[46]

$$(L)^2 \div \frac{(L)^2}{L_{c_1}} = L_{c_1}. \qquad \ldots \quad (3.47)$$

The method of spread :[47]

The polar set $\frac{(L)^2}{L_{c_1}}$ is spread so as to keep every one of its elements separate, and each such separate element is given the amount $(L)^2$. The result is that each such division is L_{c_1}.

The method of reduction :[48]

The polar set is established in form of logs (characteristics). Every one of the logs is to be removed one by one after every operation of reducing L_{c_1} from $(L)^2$.

Vīrasena asserts that by repeating this operation, the $(L)^2$ and $\frac{(L)^2}{L_{c_1}}$ were simultaneously exhausted, and whatever set is reduced each time from $(L)^2$ is the L_{c_1}.

The method of measure :[49]

$$(L)^{\frac{1}{2}}[(L)^{\frac{1}{2}}-1]+(L_{e^0}+\sum_{n=2}^{14} L_{c_n})\left[\frac{(L)^{\frac{1}{2}}}{L_{e^0}+\sum_{n=2}^{14} L_{c_n}}-1\right]$$

$$= L-(L)^{\frac{1}{2}}+(L)^{\frac{1}{2}}-(L_{e^0}+\sum_{n=2}^{14} L_{c_n})$$

$$= L_{c_1} \qquad \ldots \quad (3.48)$$

The method of reason :[50]

$$(L)^2 \div L = L \qquad \ldots \quad (3.49)$$

$$(L)^2 \div \left[L+\frac{L}{2} \right] = - L = L-\frac{L}{3} \qquad \ldots \quad (3.50)$$

The equation (3.50) has been represented geometrically in Fig. 1. It cannot be said whether it is an original Indian contribution.

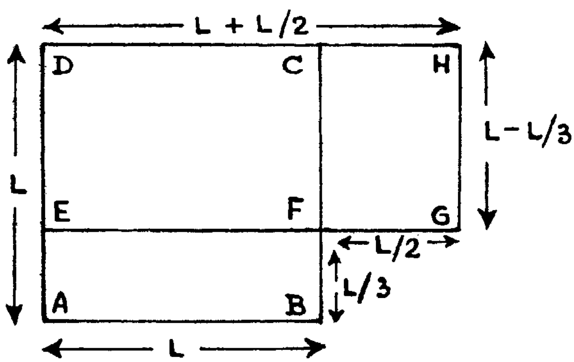

Fig. 1

Similarly it has been expressed, that

$$\frac{(L)^2}{L+\frac{L}{3}} = \frac{3}{4} L = L - \frac{L}{4} \qquad \ldots \quad (3.51)$$

The above has also been represented geometrically as in the following Fig. 2.

Here, in Fig. 1.

Fig. ABCD−Fig. ABFE = Fig. EFCD = Fig. EGHD−Fig. FGHC

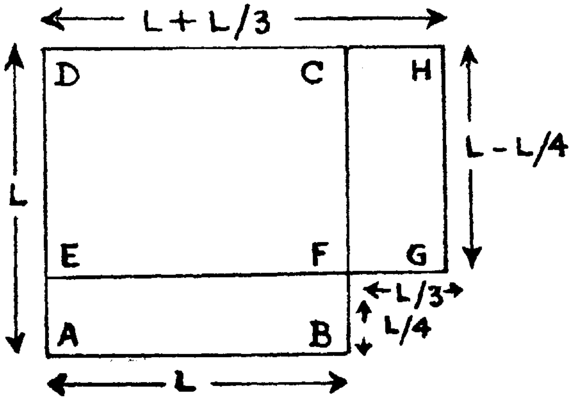

Fig. 2

In Fig. 2. Fig. ABCD = Fig. EGHD,

Fig. EFCD+Fig. ABFE = Fig. ABCD = Fig. EFCD+Fig. FGHC

$$\frac{(L)^2}{(L)+\frac{L}{\zeta}} = L - \frac{L}{\zeta+1}, \quad \ldots \quad (3.52)$$

Again,

$$\frac{(L)^2}{L+\frac{L}{\mathcal{A}}} = L - \frac{L}{\mathcal{A}+1}, \quad \ldots \quad (3.54)$$

where the author has taken only $\mathcal{A}$ in place of $\mathcal{A}+1$, and further,

$$\frac{(L)^2}{L+\frac{L}{\mathcal{A}_{au}}} = L - \frac{L}{\mathcal{A}_{au}+1} = L - \frac{L}{\mathcal{J}_{pj}} \,. \quad \ldots \quad (3.55)$$

Similarly, in case of

$$\frac{(L)^2}{L+\frac{L}{\mathcal{J}}} = L - \frac{L}{\mathcal{J}+1}, \quad \ldots \quad (3.56)$$

$\frac{L}{\mathcal{J}+1}$ has been expressed as the infinitesimal part of L.

After the above treatment Vīrasena proceeds to justify the above by quoting a few rules about fractions which were exposed by Singh in his introduction, "Mathematics of the *Dhavalā*"[51]. Then the reader is advised to follow the hinder- abstraction :[52]

$$(L)^2 \div \{L+[L_{c_1} \div (L_{c_1}+\sum_{n=2}^{14} L_{c_n})]\}$$

$$= L-[L \div (L_{e^0}+\sum_{n=2}^{14} L_{c_n})] \qquad \ldots \quad (3.57)$$

or

$$\frac{(L)^2}{L+L_{c_1} \div (L_{e^0} + \sum_{n=2}^{14} L_{c_n})}$$

$$= L- \frac{(L)}{L_{e^0} + \sum_{n=2}^{14} L_{c_n}} \qquad \ldots \quad (3.58)$$

In the above equation the equality holds good only in case the denominator of the left hand side is replaced by

$$L+ \frac{L}{L_{e^0} + \sum_{n=2}^{14} L_{c_n}-1}$$

if we follow the rules about fractions in case of sets also treated here as cardinals for which the quotient set formalism is adopted in modern set theory.

When limits are considered, we have :

$$\frac{(L)^2}{\text{L}+ \frac{\text{L}}{L_{e^0} + \sum_{n=2}^{14} L_{c_n}-1}} = \frac{(L)^2}{L+\frac{L_{c_1} + L_{e^0}+ \sum_{n=2}^{14} L_{c_n}}{L_{e^0}+ \sum_{n=2}^{14} L_{c_n}-1}}$$

$$= \frac{(L)^2}{L+ \left[\frac{L_{c_1}}{L_{e^0} + \sum_{n=2}^{14} L_{c_n}-1} + \frac{1}{L_{e^0} + \sum_{n=2}^{14} L_{c_n} -1} +1\right]} \qquad \ldots \quad (3.59)$$

It is clear that in the above equations, certain terms in the denominator of the right hand side may be ignored. Thus, 1 may be ignored in $(Le^{0}+\sum_{n=2}^{14} Lc_n-1)$ in comparison with $(Le^{0}+\sum_{n=2}^{14} L_{cn})$. Similarly

$$\frac{1}{Le^{0}+\sum_{n=2}^{14} L_{cn}-1}$$

may be neglected in comparison with

$$L+\frac{L_{c1}}{Le^{0}+\sum_{n=2}^{14} L_{cn}-1}.$$

The method of explanation :[53]

$L\div(Le^{0}+\sum_{n=2}^{14} L_{cn})$ is spread and to each of its elements is distributed equal divisional parts of L. Thus everyone of the divisional parts is $Le^{0}+\sum_{n=2}^{14} L_{cn}$. One of the divisions out of L is $Le^{0}+\sum_{n=2}^{14} L_{cn}$ and the sum of the rest of the divisions is the major part of L, given by L_{c1}.

Symbolically it may be written as

$$L\div\{L\div(Le^{0}+\sum_{n=2}^{14} Lc_n)\}=Le^{0}+\sum_{n=2}^{14} Lc_n, \qquad \dots \quad (3.60)$$

and

$$L-(Le^{0}+\sum_{n=2}^{14} L_{cn})=L_{c1}, \qquad \dots \quad (3.61)$$

where the implication of "explanation" is not clear in the above.

The method of abstraction :[54]

Abstraction is defined to be of two kinds, the lower (*adhastana*) and the upper (*uparima*). The former type is classified into three kinds : the dyadic square sequence, the dyadic cube sequence, and the dyadic cube-non-cube sequence. The upper abstraction is of three kinds : the adopted, the adopted-adopted, and the adopted-multiplier. Each of this type is further classified into those in which the former abstraction has been classified. The details are as follows :

1) *The lower abstraction* :

a) The dyadic square sequence—This type of abstraction is not possible here because $\frac{(L)^2}{L_{c1}} > L$ and hence L_{c1} cannot be obtained by dividing L by $2^{\left[\log\frac{(L)^2}{L_{c1}}\right]}$.

As we know, $(L)^2 \div L_{c1} = L \cdot \frac{L}{L_{c1}}$ and $L > L_{c1}$, therefore $\frac{(L)^2}{L_{c1}} > L$.

b) The dyadic cube-sequence

$$(L)^3 \div \left[\frac{(L)^3}{L_{c1}} L\right] = L_{c1}, \quad \ldots \quad (3.62)$$

because

$$\frac{(L)^3}{L} = (L)^2. \quad \ldots \quad (3.63)$$

One more method is worthy of attention. It has been called as the method of duplation, etc. and may be illustrated as follows :

$$(L)^3 \div L = (L)^2 \quad \ldots \quad (3.64)$$

$$(L)^3 \div 2L = (L)^2 \div 2 \quad \ldots \quad (3.65)$$

$$(L)^3 \div 3L = (L)^2 \div 3 \quad \ldots \quad (3.66)$$

....,

....,

$$(L)^3 \div L_{c1}L = (L)^2 \div L_{c1}, \quad \ldots \quad (3.67)$$

and then,

$$\frac{(L)^2}{L_{c1}} \cdot L \div (L)^3 = \frac{(L)^2}{(L)^2 \div L_{c1}} = L_{c1} \quad \ldots \quad (3.68)$$

c) Dyadic cube- non-cube sequence :

$$(L^3)^{3/2} \div \left[L \cdot \frac{(L)^2}{L_{c1}} \{(L)^3\}^{\frac{1}{2}} \right] = L_{c1} \quad \ldots \quad (3.69)$$

because $$[\{(L)^3\}^3]^{\frac{1}{2}} \div \{(L)^3\}^{\frac{1}{2}} = (L)^3, \quad \ldots \quad (3.70)$$

and $$(L)^3 \div L = (L)^2 \quad \ldots \quad (3.71)$$

as well as $$\frac{(L)^2}{(L)^2 \div L_{c1}} = L_{c1} \quad \ldots \quad (3.72)$$

2) *The upper abstraction* :

a) The dyadic square sequence; Adopted abstraction (*dvirūpa varga dhārā : gṛhīta vikalpa*) :

$$\frac{(L)^2}{(L)^2 \div L_{c1}} = L_{c1} \qquad \dots \quad (3.73)$$

and

$$\frac{(L)^2}{2^{[\log_2 (L)^2 \div Lc_1]}} = L_{c1}. \qquad \dots \quad (3.74)$$

The equation (3.74) has been explained through the following methods :

(*i*) If we suppose $\log \dfrac{(L)^2}{L_{c1}} = \log L,$ ... (3.75)

then
$$\frac{(L)^2}{L_{c1}} \div 2^{\log L} = \frac{L}{L_{c1}}. \qquad \dots \quad (3.76)$$

Therefore
$$\frac{(L)^2}{2^{\log L}} = L, \qquad \dots \quad (3.77)$$

and
$$\frac{(L)^2}{2^{\log L}} \div \frac{L}{L_{c1}} = L_{c1} \qquad \dots \quad (3.78)$$

It is not understood why Vīrasena takes the help of (3.75) in order to derive (3.76). However, it may be stipulated that he might have been led to the approximation result (3.75) by taking the logarithm of the eternal set to be the number of the processes of mediating the eternal set $\frac{(L)^2}{L_{c1}}$ in (3.76), which is $\log L$.

(*ii*) If it is supposed that

$$\log \frac{(L)^2}{L_{c1}} = \log (L)^2 \qquad \dots \quad (3.79)$$

then
$$\frac{(L)^2}{L_{c1}} \div 2^{\log (L)^2} = \frac{1}{L_{c1}} \qquad \dots \quad (3.80)$$

Further,
$$(L)^2 \div 2^{\log \frac{(L)^2}{L_{c1}}} = 1, \qquad \dots \quad (3.81)$$

and this unity so obtained, when divided by $\frac{1}{L_{c1}}$ gives L_{c1}.

Now it is important to note (3.81). $\log \frac{(L)^2}{L_{c1}}$ is to be spread into form of logs, and (L) is to be mediated till unity is attained. It is clearly obvious that Vīrasena has definitely made use of (3.79) here. In fact, we could write (3.81) as follows by virtue of (3.79) :

$$(L)^2 \div (2)^{\{\log(L^2 \div L_{c1})\}}$$

$$= (L)^2 \div 2^{\log(L)^2}$$

$$= 1 \qquad \ldots \quad (3.82)$$

(*iii*) $$(L^2)^2 \div \left[(L)^2 \cdot \frac{(L)^2}{L_{c1}} \right] = L_{c1} , \qquad \ldots \quad (3.83)$$

because $$(L^2)^2 \div L^2 = (L)^2, \qquad \ldots \quad (3.84)$$

and by virtue of (3.73).

It has also been noted that

$$\{(L)^2\}^2 \div 2^{\left\{\log\frac{(L)'}{L_{c1}}\right\}} = L_{c1} \qquad \ldots \quad (3.85)$$

Vīrasena has shown how to get the logarithm of the denominator in the left hand side of the equation (3.83). The method is as follows :

From $\{(L)^2\}^2$, we get $(L)^4$.

Then

$$(4-1) \log_2 L + \log_2 \frac{L}{L_{c1}}$$

$$= \log \frac{(L)^4}{L_{c1}} \qquad \ldots \quad (3.86)$$

Here $\log_2 \left[\frac{L}{L_{c1}}\right]$ has been called the last denominator (*bhāgahāra*). Vīrasena proceeds to illustrate the same through the logarithms to different bases. Thus :

$$\frac{(L)^2}{3^{\left\{\log_3\left(\frac{L^2}{L_{c1}}\right)\right\}}} = L_{c1}, \qquad \ldots \quad (3.87)$$

$$\frac{(L)^2}{n^{\left\{\log_n\left(L^2 \div L_{c1}\right)\right\}}} = L_{c1} \qquad \ldots \quad (3.88)$$

He further asserts that the above statement should also be made with respect to the summable, non-summable and infinite square-stations. He suggests the following method for getting the number of processes of division to be made by the divisor 2.

As $$(L)^{2n} \div \left[\frac{(L)^{2n}}{L_{c1}}\right] = L_{c1} \qquad \ldots \quad (3.89)$$

where n may be ζ, $\mathscr{A}$ or $\mathscr{J}$, therefore the number of processes of mediation would be

$$\log_2[(L)^{2^n} \div L_{c1}] \qquad \dots (3.90)$$

so that,

$$(L)^{2^n} \div 2^{\left\{\log_2 \frac{(L)^{2^n}}{Lc_1}\right\}} = L_{c1} \qquad \dots (3.91)$$

The expression (3.90) may also be expressed as

$$2^n \log_2 L - \log_2 L_{c1}\ , \qquad \dots (3.92)$$

or, approximately

$$(2^n - 1) \log_2 L. \qquad \dots (3.93)$$

(*b*) Dyadic cube sequence : The adopted upper abstraction (*aṣṭa rūpa dhārā* : *gṛhīta-uparima-vikalpa*) :

$$(L^3)^2 \div \left[\{(L)^2\}^2 \cdot \frac{(L)^2}{L_{c1}}\right] = L_{c1} \qquad \dots (3.94)$$

and

$$[(L)^3]^2 \div 2^{\left[[\log_2 \left\{((L)^2)^2 \ \frac{L^2}{Lc_1}\right\}\right]} = L_{c1} \qquad \dots (3.95)$$

Further details to be followed are the same as in the preceding cases.

(*c*) Dyadic cube-non-cube sequence : The adopted upper abstraction (*ghanā-ghana dhārā* : *gṛhita-uparima-vikapla*) :

$$[(L^3)^3]^2 \div \left[(L^2)^2 \cdot \frac{L^2}{L_{c1}} \cdot \{(L^3)^2\}^2\right] = L_{c1} \qquad \dots (3.96)$$

or

$$(L)^{18} \div 2^{\left[\log_2\left\{(L)^{18} \div L_{c1}\right\}\right]} = L_{c1} \qquad \dots (3.97)$$

In the above treatment, the sentences in the text have been abbreviated. Further details to be followed are the same as in the preceding cases.

(*d*) Dyadic square-sequence in the adopted-adopted upper abstraction (*dvirūpa vargadhārā-gṛhīta-uparima-vikalpa*):

$$(L^2)^2 \div \{(L^2)^2 \div L_{c1}\} = L_{c1}\ , \qquad \dots (3.98)$$

and

$$(L)^4 \div 2^{\left\{\log_2\left(\frac{L^4}{Lc_1}\right)\right\}} = L_{c1}, \qquad \dots (3.99)$$

where it has been stated that

$$\log_2(L^4 \div L_{c1}) = \log_2(L^4) - \log_2(L_{c1}) \qquad \dots (3.100)$$

(*e*) Dyadic cube-sequence in the adopted-adopted upper abstraction :(*Aṣṭa rūpa*: *ghanadhārā—gṛhīta-gṛhīta-uparima-vikalpa*) :

$$(L^3)^2 \div [(L^3)^2 \div L_{c1}] = L_{c1}, \quad \ldots \quad (3.101)$$

where also

$$(L^3)^2 \div 2^{\left\{\log_2 (L^6 \div Lc_1)\right\}} = L_{c1}, \quad \ldots \quad (3.102)$$

with similar details as in the preceding cases.

(*f*) Dyadic cube-non-cube sequence in the adopted-adopted upper sequence : (*ghnāghana dhārā gṛhīta-gṛhīta-uparima-vikalpa*) :

$$[\{(L^3)^3\}^{\frac{1}{2}}]^2 \div [(\{(L^3)^3\}^{\frac{1}{2}})^2 \div L_{c1}] = Lc_1 \ , \quad \ldots \quad (3.103)$$

and

$$L^9 \div 2^{\left\{\log_2 (L^9 \div Lc_1)\right\}} = L_{c1} \, , \quad \ldots \quad (3.104)$$

where the results in (3.104) have been abbreviated. Similar details as in the preceding cases be also given.

(*g*) Dyadic Square sequence in adopted multiplier-upper-abstraction: (*dvirūpa varga dhārā* : *gṛhīta-guṇakāra-uparima-vikalpa*) :

$$\{(L^2)^2\}^2 \div [\{(L^2)^2\}^2 \div L_{c1}] = L_{c1} \quad \ldots \quad (3.105)$$

and

$$L^8 \div 2^{\log_2 (L^8 \div Lc_1)} = L_{c1} \quad \ldots \quad (3.106)$$

with similar further details as in the preceding cases.

(*h*) Dyadic cube sequence in the adopted multiplier abstraction (*ghanadhārā gṛhīta-guṇakāra vikalpa*) :

$$\{(L^3)^2\}^2 \div [\{(L^3)^2\}^2 \div L_{c1}] = L_{c1} \ , \quad \ldots \quad (3.107)$$

and,

$$L^{12} \div 2^{\log_2 (L^{12} \div Lc_1)} = L_{c1} \ , \quad \ldots \quad (3.108)$$

with similar previous details.

(*i*) Dyadic cube non-cube sequence in the adopted multiplier-upper-abstraction : (*ghanāghana dhārā* : *gṛhīta guṇakāra uparima vikalpa*) :

$$[\{\{(L^3)^3\}^{\frac{1}{2}}\}^2]^2 \div \frac{[\{(L^3)^3\}^{\frac{1}{2}}]^2[\{(L^3)^3\}^{\frac{1}{2}}]^2}{L_{c1}} = L_{c1} \ , \quad \ldots \quad (3.109)$$

and in short,

$$L^{18} \div 2^{\log_2 (L^{18} \div Lc_1)} = L_{c_1} , \quad \ldots \quad (3.110)$$

with similar other details as in the preceding cases.

4. Conclusion

The above symbolic representation cannot do full justice to the various interpretations and implications of the detailed semantical approaches in the text.

It is a part of what is known as the general description (*sāmānya*) which is in relation to the control stations alone. The description which is in relation to the rummage stations is called *nirdeśa* and is in far greater details. Thus various kinds of sets have been described and manipulated in the *Dhavalā* commentary which is undoubtedly a descriptive and informative source book on sets.

As the above treatment covers several pages of the text from page 1 to page 63, most of the unimportant original Prakrit verses and sentences have been ommitted, as they do not refer to the above symbolic presentation.

References and Notes

[1] Puṣpadanta and Bhūtabali, *Ṣaṭkhaṇḍāgama* with *Dhavalā* commentaries of Vīrasena, Book 4, 1944, Amaraoti. Article of Singh, A. N., appears on pp. v-xxi.

[2] *Jaina Antiquary*, xv-2, 1949 and xvi-2, 1950, pp. 46-53 and 54-69.

[3] Jain, L. C., *On the Jaina School of Mathematics*, Calcutta, 1967, 265-292.

[4] Kundakunda's *Pañcāstikāya*; Nemicandrācārya's *Dravya Saṃgraha*; and Aklañka's *Tattvārtha Vārtikam.*

[5] Jain, G. R., *Cosmology old and new*, Lucknow, 1942. Vid. also Yati Vṛṣabha's *Tiloya Paṇṇatti*, Sholapur, Part 1, 1943Cf. Jain, L. C., Tiloya Paṇṇattī kā Gaṇita, Sholapur, 1-109, (1958).

[6] Vide. Jain, L. C., Set Theory in Jaina School of Mathematics, *Indian J. Hist. Sci.* **8.** (1973) pp.1-27.

[7] *Jamhā uvariṭṭhāṇam............anta parivuḍḍhī.* 93-94. *Pañcāstikāya*, op. cit., vv. 93-94.

[8] *Payaḍiṭṭhidi aṇubhāgappadesa........gadim janti.* 73. *Pañcāstikaya.*

[9] Mahāvīrācārya, *Gaṇita Sāra Saṃgraha*, Sholapur, introduction, 1963.

[10] *Addhāra pallachedo........sūi jagaseḍhī. Tiloyapaṇṇattī*, 131, ch. 1.

[11] Hooper, A., *Makers of Mathematics*, London, p.31, Vid.also Kaye, G. R., *Indian Mathematics*, Calcutta, 1915, p. 130.

[12] *Idi ettha vutta......bhāve vā-Saṭkhaṇḍāgama*, book 4, op. cit., p. 11. *Annairiya...... tti na.* Ibid, book 3, p. 36.

[13] *Ṇa ca edam vakkhāṇam......rūvāhiyāṇi tti gahaṇāde........*Ibid., p. 36.

[14] Cf. *Ṣaṭkhaṇḍāgama*, op. cit., Book 4, p. vii.

[15] Cf. *Ṣaṭkhaṇḍāgama*, Book 3, pp. 34 et seq., vv. 1, 2, 4. Vid. also, *Evam solasa-battīsa-causaṭṭhi ....pattam tti.* Cf. Ibid., Book 4, p. 15.

[16] Cf. Hooper, op. cit., p.183. For details of the correspondence between the logarithmic relations between sequences, vid. Nemicandrācārya's *Trilokasāra, Rajchandra Jaina Shastramala*, Bombay, 1916. A detailed article on *Mathematics of Trilokasāra* is due for publication, at Kashi.

[17] Waerden, B. L. van der, *Science Awakening*, English Translation by A. Dresden, Groningen, Holland, 1954, pp. 18, 27.

[18] Gow, J., *A Short History of Greek Mathematics*, New York, 1923, pp. 104.

[19] Heath, T., *A History of Greek Mathematics*, vols. 1 and 2, 1921, p. 125 (1).

[20] The number measure is called *Saṃkhyā Pramāṇa*. The other measure is known as simile measure (*Upamā Pramāṇa*).

[21] *Vargita-Saṃvargita* may be translated as Squared-over-Squared. The author of *Tiloyaṇṇattī* mentions similar other operation which produces still more enormously great numbers.

[22] Cf. *Ṣaṭkhaṇḍāgama*, Book, 4, pp. v-xxi. Cf. also *Tiloyapaṇṇattī kā Gaṇita*, op. cit., pp. 55-62. Vide also commentary of Ṭoḍaramala on *Gommaṭasara*.

[23] *Saṭkhaṇḍāgama*, ibid., pp. v-xxi.

[24] *Tiloyapaṇṇatti kā Gaṇita*, op. cit., pp. 55-62.

[25] Housdorff, F., *Set Theory*, 1962, p. 41, New York. Cf. also Kamke, E., *Theory of Sets*, 1950, New York, p. 58, et seq.

[26] Cantor, G., *Contribution to the Founding of the Theory of transfinite Numbers*, Illinois, 1952, p. 104, et seq. Cf. also Wilder, R. L., *The Foundations of Mathematics*, London, 1952., p. 116, et seq. Cf. also Zlot, W., *The Role of the Axiom of Choice in the Development of the abstract Theory of Sets*, Library of Congress No. Mic 57-2764, (Doctoral Thesis, Columbia University), 1957.

[27] Cf. *Trilokasāra*, op. cit., vv.53-91.Cf. also Jaini, J. L.,*Jaina Gem Dictionary*, The Central Jaina Publishing House, Arrah, 1918, pp. 149-151. Vide also Datta, B. B., 'Mathematics of Nemicandra', *Jaina Antiquary*, Arrah, vol. 1, no. 2, pp. 25-44, 1935 It appears as if various types of the sequences having well defined structures were created by application of principles of generation, stipulated by Cantor. Nemicandra refers to an untraceable work '*Vṛhaddhārā-Parikarma*'. A paper on Divergent Sequences in *Trilokasāra* is to follow in I.J.H S.

[28] *Eso savva jīva rāsido kiñ:ūṇamicchādiṭṭhirāsido va..................jahaṇṇa-parittāṇaṇta guṇamaddhāṇam gaṇ'ūṇup aṇṇae. Cf. Ṣaṭkhaṇḍāgama*, Book 3, p. 21, et seq.

The translation of the relevant verses is as follows : "The set of the souls having mythic vision is slightly less than that of all souls. How to know that it is infinite times less ? It is explained—When the logarithm of logarithm of the least perdu infinite to the base two, alongwith unity are projected into the logarithm of the least perdu infinite to the base two, the result is the logarithm of logarithm of the least infinite-infinite to the base two".

The relation (2.07) may be translated as : "When the least infinite-infinite is multiplied by the logarithm of the least infinite-infinite to the base two, the result is the set of characteristics of logarithm to the base two of the set of the once *Vargita-Samvargita*." The relation of inequality is worthy of attention.

[29] Jain, S. A., *Reality* (Translation of *Sarvārthasiddhi of Pūjyapāda*), Calcutta, 1960. Cf. vv. 5-8, ch. 1.

[30] *Ogheṇa micchāiṭṭhī davva pamāṇeṇa kewaḍiyā, aṇantā.* 2. Cf. *Ṣaṭkhaṇḍāgama*, Book 3., p. 10, v. 1, 2, 2.

[31] *Jahaṇṇa aṇantāṇaṇtādo........rāsī ghettavvo.* Ibid., p. 19.

[32] Vīrasenācārya, *Jaya Dhavalā Sahitam Kasāya Pāhuḍam*, 1, 1943 ; 2, 1951, Chaurasi Mathura. Cf. ibid., p. 39.

[33] *Tiloyapaṇṇattī kā Gaṇita*, op. cit., pp. 57 et seq.

[34] *Aṇaṇtāṇantāhi osappiṇi-ussappiṇihī ṇa avaharaṇti kāleṇa.* 3. Cf. *Ṣaṭkhaṇḍāgama*, Book 3, v. 1, 2, 3.

[35] *Aṇantāṇantāṇaṁ osappiṇi*—........*ṇa avahijjadi.* Ibid., p. 28.

[36] The following lines describe the version of Vīrasenācārya : *Kadhaṁ solasapaḍiya- appābahugaṁ ?*............*ṇa avahirijjadi tti siddhaṁ.* Ibid., p. 30 et seq.

[37] Cf. ibid., p. 30.

[38] Cf. ibid., p. 31.

[39] *Khetteṇa aṇantāṇantā logā.* 4. Cf. ibid., v. 1, 2, 4, p. 32.

[40] *Jadhā patthena java-godhūmādi rāsī*............*etthuvaujjanti gāhā-Patthena kodavena va jaha koi miṇejja savva bījāiṁ. Evaṁ miṇijjamāṇe havanti logā aṇantā du.* 22. Cf. ibid., v. 22.

[41] The following is the treatment given by Vīrasenācārya : *Kā rajjū ṇāma ? tiriya logassa majjhima vitthāro*........*puvvutta suttehi saha virohappasaṁgādo.* Cf. ibid., p. 34 et seq.

[42] *Tiṇhaṁ pi adhigamo bhāvaamāṇaṁ.* 5. Cf. ibid., v. 1, 2, 5, p. 38.

[43] *Micchāiṭṭhirāsissa pamāṇavisaye*........*bhāṇdida- hajida-viralida-avahida-pamaṇa-kāraṇa-ṇirutti-viyappehi vattaissāmo.* Cf. ibid. p. 40.

[44] *Siddhaterasa*........*micchāiṭṭhirāsipamāṇaṁhodi.* Cf. ibid., pp. 40, 41.

[45] *Savva-jīva rāsi*........*khaṇḍidaṁ gadaṁ.* Cf. ibid., p. 41.

[46] *Dhuvarāsiṇā*........*bhājidaṁ gadaṁ.* Cf. ibid., p. 41.

[47] *Dhuvarāsiṁ*........*viralidaṁ gadaṁ.* Cf. ibid., pp. 41, 42.

[48] *Taṁ ceva dhuvarāsiṁ*........*avahidaṁ gadaṁ.* Cf. ibid., p. 42.

[49] *Tassa pamāṇaṁ*........*pamāṇaṁ gadaṁ.* Cf. ibid., pp. 42, 43.

[50] *Keṇa kāraṇeṇa ?*........*kāraṇaṁ gadaṁ.* Cf. ibid., pp. 43-51.

[52] Cf. *Saṭkhaṇḍāgama,* Book 4, p. ix et seq.

[52] *Edāhi gāhāhi*........*tti ṇa saṁdeho* (?) Cf. *Ṣaṭkhaṇḍāgama,* Book 3, p. 50.

[53] *Tassa kā*........*ṇirutti gadā.* Cf. ibid., p. 51.

[54] The original details of the treatment by Vīrasenācārya are as follows : *Jo so viyappo so duviho* ....*ghaṇāghaṇaparūvaṇā gadā.* Cf. ibid., pp 52-63.