MUNĪŚVARA'S MODIFICATION OF BRAHMAGUPTA'S RULE FOR SECOND ORDER INTERPOLATION

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When the values of a function are tabulated for some discrete values of the argument, the functional values corresponding to intermediary argumental values are obtained ordinarily by linear interpolation. For greater accuracy, higher order technique is necessary.

It is known that the famous Indian Mathematician Brahmagupta (seventh century A.D.) gave a rule for second order interpolation. This yields results equivalent to what one will get by using the Newton-Stirling formula upto the same order.

Munīśvara (seventeenth century) has given a modification, which consists in applying a process of iteration and leads to better results in some cases.

The paper presents a discussion of Brahmagupta's original rule, its modification by Munīśvara, and of the example which has been worked out by the latter.

SYMBOLS :

a—the argument, circular arc measured in angular units.

$a_0, a_1, a_2, \cdots$ successive and equidistant values of a with $a_0=0$.

$D_1, D_2, D_3, \cdots$ tabulated functional differences,

$$D_1=f(a_1)-f(a_0),$$
$$D_2=f(a_2)-f(a_1), \text{ etc.}$$

$D_m=(1/2)(D_p+D_{p+1})$.

D_p = tabulated functional difference just crossed over (***bhukta-khaṇḍa***).

D_{p+1} = current tabulated functional difference (***bhogya-khaṇḍa***).

D_t = the true (*sphuṭa*) value of the current functional difference as given by Brahmagupta.

$f(a)$=functional value corresponding to the argumental value a. The function considered here is either the Indian Sine ($=R \sin a$), or the Indian Versed Sine (R vers a). So that we have $f(a_0)=0$.

h—equal (or common) arcual interval,

$$h=a_1-a_0=a_2-a_1=\ldots\ldots$$

$n=\theta/h$.

p=positive integer.

R—*Sinus totus*, radius of the circle of reference defining Sine and Versed Sine.

T—mathematically exact value of the current functional difference, so that

$$f(x+\theta)=f(x)+\theta\, T/h.$$

T_1, T_2, T_3,...successive approximations to T according to Munīśvara.

T_∞—the theoretically ultimate or limiting value in the above sequence.

$x=p.\,h$, the arc crossed over such that $f(x)$ is known.

$\triangle$=first order forward finite difference operator

$$\triangle f(a)=f(a+h)-f(a),$$

$$\triangle f(x)=D_{p+1}.$$

∇=first order backward finite difference operator

$$\nabla f(a)=f(a)-f(a-h),$$

$$\nabla f(x)=D_p.$$

θ=residual arc such that $f(x+\theta)$ is required to be found out or interpolated, θ being positive and less than h.

1. Introduction and Brahmagupta's Rule

Tabular values of the trigonometric functions $R \sin a$ and R vers a or of their differences are found in several astronomical works of ancient and medieval India. For computing the functional values corresponding to the intervening values of the argument the ordinary method used was that of linear proportion. This usual method of first order interpolation can be expressed as

$$f(x+\theta)=f(x)+(\theta/h).\ \ [f(x+h)-f(x)]$$

$$=f(x)+(\theta/h).\ \ D_{p+1}. \qquad (1)$$

For better results, more elegant techniques using second order interpolation schemes are also found in some of the Indian works of the earlier period.

One such rule is found in the works of Brahmagupta (seventh century A. D.). What he gives is equivalent to the following expression called *sphuṭa* (true) functional difference to be crossed over[1]

$$D_t=(1/2).\ (D_p+D_{p+1})-(1/2).\ (D_p-D_{p+1}).\ (\theta/h). \qquad (2)$$

Then the required result is obtained by using the relation

$$f(x+\theta)=f(x)+(\theta/h).\ D_t. \qquad (3)$$

Combining (2) and (3) and using the notation of finite difference operators, we easily get the following formulas

$$f(x+nh)=f(x)+(n/2).\ [\Delta f(x-h)+\Delta f(x)]+(n^2/2).\Delta^2 f(x-h) \qquad (4)$$

and

$$f(x+nh)=f(x)+(n/2).\ [\nabla f(x)+\nabla f(x+h)]+(n^2/2)\ \nabla^2 f(x+h) \qquad (5)$$

which can be regarded as the modern forms of Brahmagupta's second order interpolation rule using forward and backward differences respectively.

The formula (4) is a particular case (upto second order) of the more general Newton-Stirling Interpolation Formula of modern calculus of finite differences[2].

Brahmagupta's rule is found subsequently in the works of Govindasvāmin (ninth century), Vaṭeśvara (tenth century). Bhāskara II (twelfth century) and Parameśvara (fifteenth century)[3].

The general form of Brahmagupta's expression (2) will be

$$D_t=(1/2).\ [f(x+h)-f(x-h)]-(1/2).\ [2f(x)-f(x-h)-f(x+h)].\ (\theta/h)$$

which, on using Taylor Series expansions, will become

$$D_t=h\,[f'(x)+(\theta/2).\ f''(x)+(h^2/6).\ f'''(x) + (\theta h^2/24).\ f^{iv}(x)+(h^4/120).\ f^{v}(x)+\ldots] \qquad (6)$$

Now, the mathematically exact value of the current functional difference will be given by

$$T=(h/\theta).\ [f(x+\theta)-f(x)] = h\,[f'(x)+(\theta/2).\ f''(x)+(\theta^2/6).\ f'''(x) + (\theta^3/24).\ f^{iv}(x)+(\theta^4/120).\ f^{v}(x)+\ldots] \qquad (7)$$

Thus we have

$$T-D_t=-h\,\frac{(h^2-\theta^2)}{6}\,f'''(x)-\frac{\theta h\,(h^2-\theta^2)}{24}\,f^{iv}(x)+\cdots \qquad (8)$$

Since h is small and θ still smaller we may leave the subsequent terms involving higher powers of these small quantities in order to consider the sign of the R. H. S. of (8). Since the third and fourth derivatives of the Versed

Sine function are both negative, the R. H. S. of (8), presumed to be dominated by the first two terms, will be positive. And hence T will be greater than D_t.

In the case of the Sine function, we have

$$T - D_t = (h/6).\ (h^2 - \theta^2).\ [\cos x - (\theta/4).\ \sin x] \tag{9}$$

neglecting subsequent terms. So that T will be greater than D_t provided that

$$\cot x > \theta/4$$

or, *a fortiori* (since, in the first quadrant, cotangent decreases and the greatest values of θ and x are h and $90° - h$ respectively) if,

$$\tan h > h/4$$

which is always true under the conditions. Therefore, Brahmagupta's 'true' functional difference D_t may be taken to be less than the ***really true*** (or exact) functional difference.

Thus we see that, if one wants to improve Brahmagupta's expression (2), it should be modified in such a way as to yield an expression which is greater in magnitude. One such modification, found in a commentary (*circa* 1635) by Munīśvara, is discussed below.

2. Munīśvara's Modification of the Rule

Brahmagupta's rule (adopting it for a tabular interval of 10°, instead of 15°) has been given by Bhāskara II (1150 A.D.) in the *Graha-gaṇita* part (Chapter II, stanza 16) of his *Siddhānta-Śiromaṇi* and the scholiast Munīśvara (1635) in his commentary *Marīci* (= *MC*) on it, gives not only an exposition of the subject but also a modification of the rule.[6]

This modification, which is meant for achieving greater accuracy (*sūkṣmatā*), consists of applying a process of iteration (*asakṛt-karma*). The theory of the process, as gathered or based on the numerical example worked out in the *MC* (p. 134), may be outlined as follows :

We successively find the values of T_1, T_2, $\cdots$ by using (2) which can be written as

$$D_t = D_m - (\theta/2h).\ D_p + (\theta/2h).\ D_{p+1}. \tag{10}$$

The initial value is taken as

$$T_1 = D_t$$

and the subsequent values are computed by the iteration formula

$$T_{n+1} = D_m - (\theta/2h).\ D_p + (\theta/2h).\ T_n \ldots \tag{11}$$

obtained from (10). The limiting value will be obtained by making n tend to infinity. Thus we get

$$T_\infty = D_m - (\theta/2h).\ D_p + (\theta/2h).\ T_\infty \tag{12}$$

giving

$$T_\infty = [(h-\theta).\ D_p + h.\ D_{p+1}]/(2h-\theta) \tag{13}$$

Example from the MC

By applying the iteration process represented by (11), the *MC* (p. 134) works out an example of computing the Sine of 24° (which was the Indian value for the obliquity of the ecliptic) from the following (here partially reproduced) Table belonging to the *Siddhānta-Śiromaṇi, Graha-gaṇita,* II, 13 (*MC*, p 127)

TABLE I (R = 120)

a	*R Sin a*	*functianal difference*
10°	21′	21 = D_1
20°	41′	20 = D_2
30°	60′	19 = D_3
40°	77′	17 = D_4

Here

$$h = 10°,\ \theta = 4°,\ x = 20,\ p = 2\ ;$$

$$D_p = D_2 = 20\ ;\quad D_{p+1} = D_3 = 19.$$

We have, by (11),

$$T_{n+1} = 15'30'' + (1/5).\ T_n \tag{14}$$

which is the required iteration formula for finding T_n to any desired degree. However, we have noticed some calculation and printing mistakes in the *MC* values while doing the computation work ourselves. The results are shown in Table II.

Using (13), the limiting value will be

$$T_\infty = 19\ ;\ 22,\ 30$$

With this value used for T, we have

$$\text{Sin } 24° = 41 + 7\ ;\ 45 = 48\ ;\ 45.$$

Brahmagupta's rule (2) would give

$$\text{Sin } 24° = 41 + 7\ ;\ 43,\ 12 = 48\ ;\ 43,\ 12$$

while the modern value is about 48 ; 48, 30.*

*Linear interpolation yields 48 ; 36

TABLE II (T_{1+n} and $R \sin 24°$)

n		Actual Value of $T_{n+1} = 15; 30+(1/5). T_n$	Printed Text Value of T_{n+1} (*MC*, p. 134)	Value of T_{n+1} as calculated (by us) by using the printed value of T_n *each time.*
0	T_1	19 ; 18	19, 18	———
1	T_2	19 ; 21, 36	19 ; 21, 36	19 ; 21, 36
2	T_3	19 ; 22, 19, 12	19 ; 22, 22, 12	19 ; 22, 19, 12
3	T_4	19 ; 22, 27, 50. 24	19 ; 22, 28, 26, 24	19 ; 22, 28, 26, 24
4	T_5	19 ; 22, 29, 34, 4, 48	19 : 22, 29, 41, 16, 47	19 ; 22, 29, 41, 16, 48
5	T_6	19 : 22, 29, 54, 48, 57, 36	19 ; 22, 29. 56, 15, 21, 36	19 ; 22, 29, 56, 15, 21. 24
6	T_7	19 ; 22, 29, 58, 57, 47, 31, 12	19 ; 22, 29, 59, 15, 4, 19, 12	19 ; 22, 29, 59, 15, 4, 19, 12
7	T_8	19 ; 22, 29, 59, 47, 33, 30, 14. 24	19 ; 22, 29, 59, 51, 0, 51, 50, 24	19 ; 22, 29, 59, 51, 0, 51, 50, 24
8	T_9	19 ; 22, 29. 59, 57, 30, 42; 2, 52, 48	19 ; 22. 29, 59, 58, 12, 10, 22. 4, 48	19 ; 22, 29, 59, 58. 12, 10, 22, 4. 48
9	T_{10}	19 ; 22, 29, 59, 59. 30, 8, 24, 34, 33, 36	19 ; 22, 29, 59, 59, 38, 20, 34, 14, 57, 36	19 ; 22, 29, 59, 59, 38, 26, 4, 24, 57, 36
Sin 24°		48 ; 44, 59, 59. 59, 48, 3, 21, 49, 49, 26, 24	48 ; 44. 59, 59, 59, , 51, 20, 13, 45, 59, 2, 24	48 ; 44, 59, 59, 51, 20, 13. 41, 59, 2, 24
		(got by using the above value of T_{10})	(as *printed* in the text)	(got by using the *printed* value of T_{10})

Although the *MC* (p. 134) states that the technique can be used for the Versed Sine also, but its author has not worked out there any example to illustrate the process for the Versed Sine. On the other hand, we found that the process does not give satisfactory results. So I leave the matter for further discussion and investigation.

REFERENCES

[1] Gupta, R. C. Second Order Interpolation in Indian Mathematics etc.. *Indian J. Hist. Sci.*, Vol. 4. (1969), p. 88

The Sanskrit couplet on which Brahmagupta's rule (for equal knots) is based and for which we have quoted several printed references in the above paper, is found in the *Uttara* part of the *Khaṇḍa-Khādyaka* with Utpala's commentary recently (1970) edited by Bina Chatterjee (see Vol. II, p. 177).

[2] Whittaker, E. and Robinson, G. *The Calcuuls of Observations*, Blackie, London, 1965 ; p. 38.

[3] Gupta, R. C. *Op.* cit., pp. 88-95. The rule is also found in the *Siddhānta Sārvabhauma*, II, 110 of Muīśvara (1646) ; See M. Thakkura's edition, Part I, p. 172 (Govt. Sanskrit College. Benares, 1932).

[4] *Siddhānta Śiromaṇi Grahagaṇita*, with the *MC* etc., edited by K. D. Joshi. part II, pp. 133-143, B. H. U. Varanasi, 1964.