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A RATIONALE OF BRAHMAGUPTA'S METHOD OF SOLVING $ax+c=by$

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Indian Scholar Brahmagupta (*c*.628) perhaps used the method of continued fraction to find out the integral solution of the indeterminate equation of the type $ax+c=by$. But for the solution of $ax+c=by$, Pṛthudakasvāmī's commentary is helpful. This paper presents the original Sanskrit verses from Brahmagupta's *Brāhmasphuṭa Siddhānta*, its English translation with modern interpretation.

1. INTRODUCTION

Bag[1] discussed the method of solution of indeterminate equation of the type $by=ax\pm c$ in ancient and medieval India. In his paper he discussed the method of Āryabhaṭa I and Bhāskara I in details. Then without going into details he said "The method was subsequently discussed by Brahmagupta (*c*. 628 A. D.), Pṛthudakasvāmī (*c*. 850), Śrīpati (*c*. 1039), Govindasvāmī (*c*. 850), Mahāvīra (*c*. 850) with no improvement." Finally he drew the conclusion "Indian scholars had a more or less distinct idea about the application of continued fraction and used the tool $p_nq_{n-1}-q_np_{n-1}=\pm 1$ for the solution of $by=ax\pm c$ according as n is even or odd."

Bag never went through the detailed study of Brahmagupta's method for obtaining general solution of the linear indeterminate equation of the type $by=ax+c$. In this paper, we are going in details to study Brahmagupta's method for the solution of the equation $ax+c=by$ and then show that until and unless we use Pṛthudakasvāmī's commentary in this respect, it is impossible to draw the conclusion that Brahmagupta had an idea about the result $p_nq_{n-1}-q_np_{n-1}=(-1)^n$ of the continued fraction.

2. RULE

Brahmagupta (628 A. D.)[2] gave the following rule in his *Brāhmasphuṭa Siddhānta :*

adhikāgrabhāgahārādūnāgra cchedabhājitaccheṣam
yattat parasparahṛtaṃ labdhamadho'dhoḥ pṛthak sthāpyam ‖ 3 ‖

śeṣaṃ tathaiṣṭaguṇitaṃ yathāgrayorantareṇa saṃayuktam
śudhyati guṇakaḥ sthāpyo labdhaṃ cāntyādupāntyaguṇaḥ ‖ 4 ‖

svordhbo'ntyayuto' grānto' hīnāgra ccehedabhājitaḥ śeṣaṃ
adhikāgraccedahṛtamadhikāgrayutaṃ bhavatyagraṃ ‖ 5 ‖

Datta and Singh translate these verses as follows :

'What remains when the divisor corresponding to the greater remainder is divided by the divisor corresponding to the smaller remainder—that (and the latter divisor) are mutually divided and the quotients are severally set down one below the other. The last residue (of the reciprocal division after an even number of quotients has been obtained) is multiplied by such an optional integer that the product being added with the difference of the (given) remainders will be exactly divisible (by the divisor corresponding to that residue). That optional multiplier and then the (new) quotient just obtained should be set down (underneath the listed quotients). Now proceeding from the lowermost number (in the column), the penultimate is multiplied by the number just above it and then added by the number just below it. The final value thus obtained (by repeating the above process) is divided by the divisor corresponding to the smaller remainder. The residue being multiplied by the divisor corresponding to the greater remainder and added to the greater remainder will be the number in view.'

He further observes[8]

ebaṃsameṣu biṣameṣbṛaṇāṃ dhanamṛṇaṃ yaduktaṃ tat
ṛṇadhanayorbyastatbaṃ guṇyaprakṣepayoḥ kāryam 13

"Such is the process when the quotients (of mutual division) are even in number. But if they be odd, what has been stated before as negative should be made positive or as positive should be made negative."

3. Rationale of the Method

Let a=*adhikāgrabhāgahāra* (=divisor corresponding to the greater remainder), b=*ūnāgraccheda* (=divisor corresponding to the smaller remainder), c= *agreyorantareṇa* (=difference of the two remainders). Now according to the translation, we have to consider the equation of the type $ax+c=by$.

Now $$y=a/b+c/b \qquad (1)$$

$$
\begin{array}{l}
b)\ a\ (a_1 \\
\quad \overline{\cdots} \\
\quad r_1)\ b\ (a_2 \\
\qquad \overline{\cdots} \\
\qquad r_2)\ r_1\ (a_3 \\
\qquad\quad \overline{\cdots} \\
\qquad\quad r_3)\ r_2\ (a_4 \\
\qquad\qquad \overline{\cdots} \\
\qquad\qquad r_4)\ r_3\ (a_5 \\
\qquad\qquad\quad \overline{\cdots} \\
\qquad\qquad\quad r_5)\ r_4\ (a_6 \\
\qquad\qquad\qquad \overline{\cdots} \\
\qquad\qquad\qquad r_6)\ \cdots
\end{array}
$$

Here

$$
\begin{aligned}
a &= ba_1+r_1 \\
b &= r_1a_2+r_2 \\
r_1 &= r_2a_3+r_3 \\
r_2 &= r_3a_4+r_4 \\
r_3 &= r_4a_5+r_5
\end{aligned}
$$

Case I. When even number of (partial) quotients is considered : Let the number of (partial) quotients be four. Consider the optional number t_1. According to the rule, we have

$$\frac{r_4t_1+c}{r_3}=k_1 \tag{3}$$

Consider the table

a_1	a_1L+s_2	$=U\ (=y)$
a_2	$a_2s_2+s_1$	$=L\ (=x)$
a_3	$s_1a_3+t_1$	$=s_2$
a_4	$a_4t_1+k_1$	$=s_1$
t_1	t_1	
k_1		

Here

$$
\begin{aligned}
s_1 &= a_4t_1+k_1 \\
&= a_4\,[(r_3k_1-c)/r_4]+k_1 \\
&= (a_4r_3k_1-a_4c+k_1r_4)/r_4 \\
&= [k_1(a_4r_3+r_4)-a_4c]/r_4 \\
&= (k_1r_2-a_4c)/r_4. \qquad (4)
\end{aligned}
$$

$$
\begin{aligned}
s_2 &= a_3 s_1 + t_1 \\
&= a_3\left(\frac{k_1 r_2 - a_4 c}{r_4}\right) + \frac{k_1 r_3 - c}{r_4}\left[\because\ t_1 = \frac{k_1 r_3 - c}{r_4} \text{ from } 3\right] \\
&= \frac{k_1(a_3 r_2 + r_3) - c(a_3 a_4 + 1)}{r_4} \\
&= \frac{k_1 r_1 - c(a_3 a_4 + 1)}{r_4}. \qquad (5)
\end{aligned}
$$

$$
\begin{aligned}
L &= a_2 s_2 + s_1 \\
&= a_2\left[\frac{k_1 r_1 - c(a_3 a_4 + 1)}{r_4}\right] + \frac{k_1 r_2 - a_4 c}{r_4} \\
&= \frac{k_1(a_2 r_1 + r_2) - c(a_2 a_3 a_4 + a_2 + a_4)}{r_4} \\
&= \frac{k_1 b - c(a_2 a_3 a_4 + a_2 + a_4)}{r_4} \\
&= \frac{k_1 b - c q_4}{r_4}. \qquad (6)
\end{aligned}
$$

$$
\begin{aligned}
U &= a_1 L + s_2 \\
&= a_1\left[\frac{k_1 b - c(a_2 a_3 a_4 + a_2 + a_4)}{r_4}\right] + \frac{k_1 r_1 - c(a_3 a_4 + 1)}{r_4} \\
&= \frac{k_1(a_1 b + r_1) - c[a_1(a_2 a_3 a_4 + a_2 + a_4) + (a_3 a_4 + 1)]}{r_4} \\
&= \frac{k_1 a - c[a_1(a_2 a_3 a_4 + a_2 + a_4) + (a_3 a_4 + 1)]}{r_4} \\
&= \frac{k_1 a - c p_4}{r_4}, \qquad (7)
\end{aligned}
$$

where p_4/q_4 is the fourth convergent of the continued fraction

$$a_1 + \frac{1}{a_2 +}\frac{1}{a_3 +}\frac{1}{a_4 +}\cdots$$

Now

$$U/L = \frac{k_1 a - c p_4}{k_1 b - c q_4} \qquad (8)$$

Here

$$p_5/q_5 = a/b, \qquad \therefore\ p_5 = a,\ q_5 = b.$$

Consider

$$\begin{aligned} p_5L-q_5U &= p_5(k_1b-cq_4)-q_5(k_1a-cp_4) \\ &= a(k_1b-cq_4)-b(k_1a-cp_4) \\ &= abk_1-acq_4-abk_1+bcp_4 \\ &= c(bp_4-aq_4) \\ &= c(q_5p_4-p_5q_4) \\ &= -c(p_5q_5-q_5p_4) \\ &= +c. \end{aligned} \tag{9}$$

We have taken $L=x$, $U=y$.

Therefore (9) becomes

$$ax-by=c$$

or,

$$ax-c=by \tag{10}$$

which is not the original form $ax-c=by$.

Case 2. When the number of quotients is odd. Let the number of quotients be five, and let the optional number be t_2, then according to the rule, we have

$$\frac{r_5t_2-c}{r_4}=k_2 \text{ (say)}. \tag{11}$$

Consider the table

a_1	La_1+s_3	$=U(=y)$
a_2	$s_3a_2+s_2$	$=L(=x)$
a_3	$s_2a_3+s_1$	$=s_3$
a_4	$s_1a_4+t_2$	$=s_2$
a_5	$t_2a_5+k_2$	$=s_1$
t_2	t_2	
k_2		

Now proceed exactly as in the Case 1. We have

$$L=a_2s_3+s_2=(k_2b+cq_5)/r_5$$

$$U=a_1L+s_3=(k_2a+cp_5)/r_5.$$

Now,

$$U/L=(k_2a+cp_5)/(k_2b+cq_5) \tag{12}$$

Here

$$p_6/q_6=a/b \qquad \therefore p_6=a,\ q_6=b.$$

Consider,

$$
\begin{aligned}
p_6 L - q_6 U &= p_6\,(k_2 b + c q_5) - q_6\,(k_2 a + c p_5) \\
&= abk_2 + acq_5 - abk_2 - bcp_5 \\
&= c\,(aq_5 - bp_5) \\
&= c\,(p_6 q_5 - q_6 p_5) \\
&= c. \qquad (13)
\end{aligned}
$$

We have taken $L=x$, $U=y$.

Therefore (13) becomes

$$p_6 x - q_6 y = c$$

or, $$ax - by = c$$

or, $$ax - c = by \qquad (14)$$

which is not the original form $ax+c=by$.

Therefore from (10) and (14) we have arrived at some mystery.

What are those mysteries ?

The mysteries are regarding the direction for dividing the divisor corresponding to the greater remainder by the divisor corresponding to the smaller remainder. In this respect, the commentator Pṛthudakasvāmī (*c.* 860)[4] observes that it is not absolute, rather optional ; so that the process may be considered in the same way by starting with the division of the divisor corresponding to the smaller remainder by the divisor corresponding to the greater remainder. But in this case of inversion of the process, he continues, the difference of the remainders must be negative.

(I am not able to collect the original extract from Pṛthudakasvāmī's version).

That is to say, the equation $by=ax+c$ can be solved by transforming it first to the following form

$$ax = by - c. \qquad (15)$$

Proceeding exactly as before, we arrive at the equation

$$by - c = ax \text{ which is (15).}$$

This shows that Brahmagupta knew the solution of $ax+c=by$ which involved the knowledge of continued fraction $p_n q_{n-1} - q_n p_{n-1} = (-1)^n$. For solution of $by=ax+c$ Pṛthudaka's suggestion is helpful.

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REFERENCES

1 Bag, A. K., The method of integral solution of indeterminate equations of the type $by=ax\pm c$ in ancient and medieval India. *Indian Journal of History of Science*, Vol. 12, No. 1, 1977, p. 1-16.

2 *Brāhmasphuṭa Siddhānta* of Brahmagupta. Edited by R. S. Sarma. Chapter 18 Verses 3-5.

3 Ibid—verse 13.

4 *History of Hindu Mathematics* (a source book). Part 2, 1938, Motilal Banarsi Das, P. 102.