Indian Journal of History of Science, **21**(2) : 131—139 (1986)

MĀDHAVACANDRA'S AND OTHER OCTAGONAL DERIVATIONS OF THE JAINA VALUE $\pi = \sqrt{10}$

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(*Received 28 May 1985*)

$\sqrt{10}$ was one of the approximate values of π used in ancient and medieval times especially in Jaina works. K. Hunrath derived it from a dodecagon a century ago, and G. Chakravarti from an octagon about fifty years ago. An ancient derivation given by Mādhavacandra (*c.* 1000 A.D.) in his sanskrit commentary on *Tiloyasāra* of Nemicandra (*c.* 975 A.D.) has been examined in detail especially in the light of expositions given by Chakravarti and Āryikā Viśuddhamatī recently. Some new and more plausible interpretations are advanced here regarding derivation of $\pi = \sqrt{10}$ based on octagons. Use of the process of averaging is also illustrated.

INTRODUCTION

It is well known that an ancient Indian rule for finding the perimeter (or circumference) p of a circle of diameter d can be expressed by the formula

$$p = \sqrt{10d^2} \qquad \ldots \quad (1)$$

This rule is found used or adopted especially in the early Jaina canonical and other works.[1] Rules equivalent to (1) are also found in non-Jaina Indian as well as foreign works.[2]

Several derivations of (1) have been suggested. According to Colebrooke (1755-1837),[3] Brahmagupta (*c.* 628 A.D.) is said to have obtained the value $\pi = \sqrt{10}$ by "inscribing in a circle of unit diameter regular polygons of 12, 24, 48, and 96 sides and calculating successively their perimeters which he found to be $\sqrt{9.65}$, $\sqrt{9.81}$, $\sqrt{9.86}$, $\sqrt{9.87}$, respectively, and to have assumed that as number of sides is increased indefinitely, the perimeter would approximate to $\sqrt{10}$". Hankel (1873) suggested the same method by taking $d = 10$ (instead of unity) but this explanation is considered doubtful by Hobson.[4]

Hunrath (*c.* 1883)[5] first calculated the arrow (= height of the segment) corresponding to the arc of a sixth part of the circumference of the circle as

$$h_6 = d(2-\sqrt{3})/4. \qquad \ldots \quad (2)$$

Then he took the approximate value 5/3 for $\sqrt{3}$ thereby getting

$$h_6 = d/12 \qquad \dots \quad (3)$$

and so the side of the inscribed dodecagon would be given by

$$\begin{aligned} s_{12}^2 &= h_6^2 + (1/4).s_6^2 \\ &= (d/12)^2 + (1/4).(d/2)^2 \\ &= 10d^2/144. \end{aligned}$$

(subscripts here denote the number of sides of the related inscribed regular polygon) Hence, finally (but approximately)

$$p^2 = (12\ s_{12})^2 = 10d^2$$

which is equivalent to (1). Hunrath's method is considered to be "most plausible" by Sarasvati Amma[6] (but see *Section 3* below for a comment on this method).

Another method based on dodecagon has been recently suggested by Afzal Ahmad[7] but it is unsatisfactory since it chooses an arbitrary denominator in approximating $\sqrt{3}$.

However, Mādhavacandra Traividya (*c.* 1000 A.D.), an ancient Jaina writer, gave a different derivation which is based on a polygon of only 8 sides (instead of 12 and more employed in above methods). We discuss below in detail the various methods based on considerations of octagons.

2. Mādhavacandra's Derivation

Nemicandra, a famous *Digambara* Jaina author (*c.* 975 A.D.) composed his *Tiloyasāra* (Sanskrit, *Trilokasāra*) in Prakrit. Its *gāthā* 96 contains the formula (1) as[8]

विक्खंभवग्गदहगुणकरणी वट्टस्स परिरयो होदि।

(विष्कम्भवर्गदशगुणकरणिः वृत्तस्य परिधिः भवति।)

"The square root of ten-times the square of the diameter becomes the circumference of the circle".

Mādhavacandra, pupil of Nemicandra, in his Sanskrit commentary on the *Trilokasāra* gives the derivation (*vāsanā*) of (1) under the above *gāthā* in the following words :[9]

......*Ekayojana-vṛtta-kṣetraṃ tatpramāṇena caturasraṃ kṛtvā bhuja-koṭyoḥ kṛtyoḥ parasparaṃ guṇayitvā* '*vivi* 1 *vivi* 1 *samāse vi vi* 2' *karṇakṛtiḥ tasyāmardhitāyāṃ* *dvitīyāṃśaḥ tasminnardhite (punarapyardhitāyāṃ) caturthāṃśa, tasminnardhite aṣṭamāṃśaṃ khaṇḍaṃ, tatraikakhaṇḍaṃ gṛhītvā bhujakoṭyoḥ dvābhyāṃ samānachedena melanaṃ kṛtvā ekakhaṇḍasya etāvati phale aṣṭakhaṇḍasya kim. Vargarāśerguṇakārabhāgaharau vargātmakau bhavata iti nyāyena icchāṅkaḥ vargarupeṇa guṇakāro bhavati. Tayorguṇakārabhāgahārayorvajrāvavartane daśaguṇite viṣkambhavāsanā bhavati.*

"(Take) a circle of diameter one *yojana* and draw a square of the same dimension. By mutual (or self) multiplication obtain the squares of the base (horizontal side, say) and upright (perpendicular side) $1dd$ and $1dd$; adding which we get $2d^2$, the square of the diagonal. Halving (each d) in it we get (the square of) half the diagonal, and by again halving, we get (the square of) the fourth part (of the diagonal) ; and by (once more) halving, we get (the square of) the eighth part of the diagonal.

"Now take one of the (eight arcual) segment, and by reducing (the squares of) the *bhuja* ($= 2d^2/16$) and *koṭi* ($= 2d^2/64$) to a common denominator, add them (to get $10d^2/64$). If this is the result for one segment (*khaṇḍa*), what will it be for the eight segments ? By the rule (*nyāya*) that when a (to be operated) quantity is in square form, its multiplier and divisor should also be in the square form, the desired (proportionality) multiplier here should be in square form (*i.e.* $8^2 = 64$). So that by mutual (or cross) cancellation (in $64 \times 10d^2/64$), the result will be $10d^2$. This gives the derivation of the above rule (starting with the word) *viṣkambha*".

From this almost literal translation of Mādhavacandra's passage, it is seen that many points in his derivation need explanation especially because he himself did not give any accompanying diagram which would have clarified the doubts. This situation has given rise to various interpretations by scholars. In the following sections we critically examine some of these interpretations.

3. G. Chakravarti's Computations

More than 50 years ago Chakravarti[10] found that Mādhavacandra's method consisted in equating the perimeter of the (inscribed regular) octagon to the circumference of the circle. In other words the arc WmR (see the figure) was taken equal to the chord WR which is a side of the octagon. To find this Chakravarti gave the following calculations :

$$WY = OW/\sqrt{2} = d/2\sqrt{2}. \qquad \cdots \quad (4)$$

This is what Mādhavacandra called *bhuja* for one *khaṇḍa* (segment or portion) and gave

$$(bhuja)^2 = 2d^2/16 \qquad \cdots \quad (5)$$

which is clearly equal to $(WY)^2$ exactly.

However, Chakravarti took the value

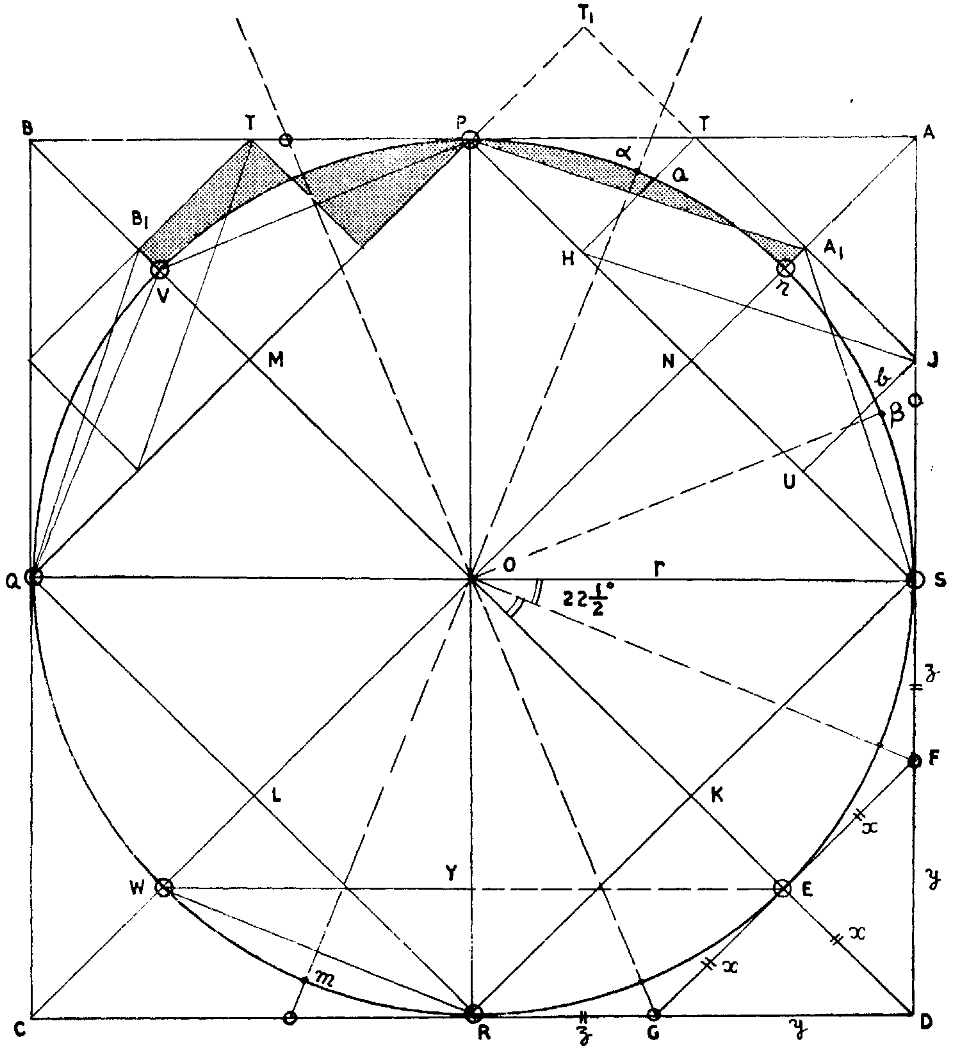
$$\sqrt{2} = 1+(1/3) = 4/3 \qquad \ldots \quad (6)$$

which he got from the *Śulbasūtra* approximation

$$\sqrt{2} = 1+(1/3)+(1/3.4)-(1/3.4.34) \qquad \ldots \quad (7)$$

whose first two terms are believed to be based on the formula (which itself is based on a sort of linear interpolation)[11]

$$\sqrt{a^2+x} = a+x/(2a+1). \qquad \ldots \quad (8)$$



By using (6), Chakravarti then got, from (4),

$$WY = 3d/8$$

or $$(WY)^2 = 9d^2/64 \quad \dots \quad (9)$$

against the mathematically exact value (5) given by Mādhavacandra. Chakravarti then calculated

$$RY = OR - OY = (d/2) - WY = d/8$$

so that

$$(RY)^2 = d^2/64 \quad \dots \quad (10)$$

while Mādhavacandra took

$$(koṭi)^2 = 2d^2/64. \quad \dots \quad (11)$$

However, the final results will be the same since

$$(WR)^2 = (WY)^2 + (RY)^2 = (9d^2/64) + (d^2/64) = 10d^2/64$$

and also, from (5) and (11),

$$(bhuja)^2 + (koṭi)^2 = (8d^2/64) + (2d^2/64) = 10d^2/64.$$

It must be noted that although Mādhavacandra's *bhuja* gives the exact value of WY, his value of *koṭi* does not represent the exact or true value of RY which is, otherwise, given by

$$RY = (d/2) - (d/2\sqrt{2}) = d(2 - \sqrt{2})/4 \quad \dots \quad (12)$$

Of course Chakravarti's values of WY and RY, given by (9) and (10), are both approximate. Any way, if WY is regarded *bhuja* in the right angled ΔWYR, then RY must be called *koṭi* therein. Now Mādhavacandra's value of his *koṭi* also represents the length of half of OY or OL or CL, and WL is equal to YR. We may therefore say that he made the practically sound assumption that W is the middle point of CL. However, there seems to be another reason as to why Mādhavacandra took his *koṭi* equal to half of his *bhuja*. We have, by using the true values (4) and (12,),

$$YR/OY = YR/WY = \sqrt{2} - 1, \text{ exactly.} \quad \dots \quad (13)^*$$

But according to the usual Jaina formula (of the binomial theorem type and based on completing the square, say)[12]

$$\sqrt{a^2 + x} = a + (x/2a) \quad \dots \quad (14)$$

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*This also follows directly from the fact that

$$OY + YR = OR = OW = \sqrt{2}.OY$$

$$i.e.\ YR = \sqrt{2}.OY - OY = (\sqrt{2} - 1)\, OY$$

so that,

$$\sqrt{2} = \sqrt{1^2+1} = 1+1/2 = 3/2 \qquad \ldots (15)$$

and thus, by (13)

$$YR/OY = 1/2 = WL/OL = WL/CL. \qquad \ldots (16)$$

That is, YR is half of WY or WL is half of LR. In other words *koṭi* is half of *bhuja.* And this theoretical basis and practical interpretation seem to be quite plausible (see *Section 4* below, for other interpretations). We have thus distinguished Mādhavacandra's method based on (14) from that of Chakravarti based on (8). It may be pointed out that Hunrath's value 5/3 for $\sqrt{3}$ (see *Section 1* above) is also based on the non-Jaina formula (8). To emphasize the contrast between the calculations of Mādhavacandra and Chakravarti we see that the value of $(WY/RY)^2$ is 4 according to the former but 9 according to the latter, while the correct value is, from (4) and (12), given by

$$(WY)^2/(RY)^2 = 3+2\sqrt{2} \approx 5.8.$$

So Mādhavacandra's value is better. It should also be noted that although point Y trisects OR according to (16) in conformity with Mādhavacandra's values, he did not take the simplified values

$$YR = OR/3 = d/6$$

and $$OY = 2.OR/3 = 2d/6 = WY$$

as these would have led to

$$(WR)^2 = (2d/6)^2+(d/6)^2 = 10d^2/72$$

instead of the desired value $10d^2/64$.

Finally, one more thing may be pointed out. The method of *inscribed* polygon, when correctly followed, should lead to a value of π which is *less* than the actual value ; but here we are getting $\sqrt{10}$ which is greater than the true value of π. The reason is that in finding the length of the side of the octagon both Mādhavacandra and Chakravarti overestimated it.

4. Āryikā Viśuddhamatī's Exposition

In her recent translation and exposition of Mādhavacandra's derivation, Viśuddhamatī[13] has correctly given the values of the squares of the bjuja and *koṭi* as mentioned in the commentary by the former and represented by (5) and (11). But from the diagrams accompanying her exposition it seems that she has taken the rectangle $THUJ$ as representing an *aṣṭamāṃśa* ("eighth part") which has also been drawn as shown separately. No doubt, the dimensions (*i.e.* the two mutually perpendicular sides HU and UJ) of this rectangle are the same as those given by Mādhavacandra. There are two difficulties in accepting this interpretation. Firstly, the arc anb contained in it is not the eighth part of the circumference of the circle (the eighth part is correctly given by the arc $\alpha n\beta$, instead of anb). Secondly, the practical equality of the arcual eighth part (anb or even $\alpha n\beta$) to the diagonal HJ of the rectangle is not clear from the diagram. So, this interpretation cannot be considered satisfactory, although it does not theoretically affect the derivation.

To remove the above difficulties, I suggest that Mādhavacandra's ***bhuja*** be taken to represent the side PN ($= HU$) and his *koṭi* be taken to represent the upright or perpendicular side $NA_1 (= UJ)$ in the right-angled ΔPNA_1 (or the rectangle PNA_1T_1). Here, the enclosed arc *Pan* is exactly equal to the eighth part of the circumference of the circle and the upright NA_1 is also the eighth part of the diagonal AC. Moreover, the equality of the resulting hypotenuse (or diagonal) PA_1 to the overlapping (or crossing) arc *Pan* seems to be a practical approximation to the eyes. Of course, the final result in this new interpretation will be the same as found by Mādhavacandra (and Viśuddhamatī, and even Chakravarti) since

$$PA_1^2 = PN^2 + NA_1^2 = (RN)^2 + (NA/2)^2$$

$$= (bhuja)^2 + (koṭi)^2 = (8d^2/64) + (2d^2/64) = 10d^2/64.$$

And so

$$p^2 \approx (8.PA_1)^2 = 10d^2, \text{ as desired.}$$

5. Applying the Process of Averaging

The process of averaging is known to be a popular and useful ancient technique especially when the exact result or derivation was unknown or difficult.[14] Even in matters of circling the square or squaring the circle, averaging has been suggested as an explanation of some of the Indian rules.[15] Here we shall confine to derivations based on considerations of octagons only.

From the figure we have

$$x = ED = (\sqrt{2}-1)r,$$

$$y = FD = \sqrt{2}x = (2-\sqrt{2})r,$$

$$z = FS = r-y = (\sqrt{2}-1)r = x.$$

These results also follow by using trigonometry, since $2x$ or $2z$ is the side of the circumscribed regular octagon and, from Δs OEF and OSF

$$x = z = r \tan (45/2)^\circ = (\sqrt{2}-1)r,$$

but we confine to more elementary and primitive methods and approach. Now the perimeter of the circumscribed octagon

$$= 8x+8z = 16x,$$

so that

$$\pi = p/2r < 16x/2r = 8(\sqrt{2}-1). \qquad (17)$$

Therefore,

$$\pi^2 < 64(3-2\sqrt{2}) < 11.$$

On the other hand by considering the inscribed regular octagon and using (4) and (12), we have

$$s_8^2 = WR^2 = (2-\sqrt{2})r^2,$$

Now,

$$2\pi r > 8.s_8$$

or,

$$\pi^2 > 16\,(2-\sqrt{2})$$

$$= 9+(23-16\sqrt{2})$$

from which it follows that

$$\pi^2 > 9.$$

Hence we have

$$9 < \pi^2 < 11,$$

and so by averaging

$$\pi^2 = 10$$

as desired and implied in (1).

Instead of perimeters, we may consider the areas of the two octagons. We see that the area of the circumscribed octagon

$$= \text{square } ABCD - 4.\Delta GDF$$

$$= (2r)^2 - 4.(y^2/2) = 8\,(\sqrt{2}-1)r^2.$$

So that

$$\pi r^2 < 8(\sqrt{2}-1)r^2$$

giving the same inequality as (17), and hence here also

$$\pi^2 < 11.$$

On the other hand, let us approximate the area of the circle by the square *PQRS* plus the four rectangles of the type *THUJ* on the four sides. From the shaded areas in the octant POB_1 we see that the area left-out is more than the extra area included.

Thus,

$$\pi r^2 > \text{square } PQRS + 4 \times (\text{rectangle } THUJ)$$
$$= 6 \times (\text{square } OMPN)$$
$$= 6 \times r^2/2 = 3r^2.$$

So that

$$\pi^2 > 9$$

Hence we get, again

$$9 < \pi^2 < 11$$

and the desired result follows by averaging as before.

References and Notes

[1]Gupta, R. C., Circumference of the Jambūdvīpa in Jaina Cosmography, *Indian Journal of History of Science*, **10**, 38–46, 1975.

[2]——, Some Ancient Values of *Pi* and Their Use in India, (Glimpses of Ancient Indian Mathematics, No. 13), *The Mathematics Education*, **9**, Sec. B, 1–5, 1975; and "The Jaina Value of π and Its Transmission Abroad", Paper presented at the International Seminar on Jaina Mathematics and Cosmology, Hastinapur (Meerut), April 26–28, 1985 and to be published in its proceedings.

[3]Quoted by L. C. Jain in his "Mathematics of the Tiloyapaṇṇatti" (Hindi), p. 20, contained in *Jambūdivapaṇṇatti-saṃgaho*, Sholapur, 1958. Also see *Algebra with Arithematic and Mensuration from the Sanskrit of Brahmagupta and Bhaskara*, translated by H. T. Colebrooke (London, 1817); reprinted Martin Sändig OHG, Wiesbaden, 1973, p. 87.

[4]Hobson, E. W., *Squaring the Circle* (1913), reprinted, Chelsea Publishing Co., N.Y. 1969, p. 23. Hankel's work has been quoted earlier by M. Cantor, *Vorlesungen über Geschichte der Mathematik*, Vol. I (3rd edn. 1907), reprinted, Johnson Reprint Corporation, New York, 1965, p. 647.

[5]Quoted by Cantor, *Vorlesungen* etc. (see ref. No. 4 above), Vol. I, p. 648.

[6]Sarasvati Amma, T. A., *Geometry in Ancient and Medieval India*, Motilal Banarsidass, Delhi, 1979, p. 65.

[7]Ahmad, A., The Vedic Principle for Approximating Square Root of Two, *Gaṇita Bhāratī*, **2**, 16–19, 1980.

[8]*Trilokasāra* with the Sanskrit Commentary of Mādhavacandra and Hindi Commentary of Āryikā Viśuddhamatī, edited by R. C. Jain Mukhtar and C. P. Patni, Shri Mahavirji (Rajasthan), *Vīranirvāṇa* year 2501 (= 1975 A.D.), p. 88 (Śivasāgara Granthamālā, No. 6).

[9]*Ibid.*, p. 89.

[10]Chakravarti, G., On the Earliest Hindu Methods of Quadratures, *Jour. Dept. Letters* (Calcutta Univ.), **24**, article no. 8, p. 28, 1934.

[11]Gupta, R. C., Baudhāyana's Value of $\sqrt{2}$, (Glimpses of Ancient Indian Mathematics, No. 3), *The Mathematics Education*, **6**, Sec. B, 77–79, 1972.

[12]——, Circumference of the Jambūdvīpa etc. (see ref. No. 1 above), pp. 42–43.

[13]*Trilokasāra* (see ref. 8 above), pp. 88–92, and Appendix, p. 7.

[14]Gupta, R. C., The Process of Averaging in Ancient and Medieval Mathematics, *Gaṇita Bhāratī*, **3**, 32–42, 1981.

[15]Chakravarti, *op. cit.* (ref. 10), 24–26.