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ARITHMETICAL RATIO OF DIAMETER TO ITS CIRCUMFERENCE OF A CIRCLE WITH SPECIAL REFERENCE TO JĀME-I-BAHĀDUR KHĀNĪ

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From ancient times Indian mathematicians tried to find better approximations of the ratio of the diameter to the circumference of a circle, and gave expressions and values without describing the method of derivation. But in later part of medieval period, mathematicians realised the need for geometrical methods for computation of this ratio. Ghulām Ḥusain Jaunpūrī (1790–1862 A.D.), a famous mathematician and astronomer of his time, lays down an interesting method for this purpose.

In the early works of Indian mathematicians it is found that the circumference of the circle is "thrice its diameter". *Matsya Purāṇa* says, "the circumference is thrice the diameter". The *Āditya Purāṇa* says, after it has mentioned the breadth of the *dvīpas i.e.* the islands and of their surrounding sea, that "the circumference is thrice the diameter".[1] The early Jaina (500—300 B.C.) works give also the same rules. Indeed, the approximation of this ratio by 3 is common in ancient mathematical texts.[2] But later on mathematicians came to know of numbers other than integers and some adopted this ratio as $\sqrt{10}$ and this value was accepted upto the time of Bhāskara (1150 A.D.) although Āryabhaṭa used more accurate value. In this paper we shall critically compare the ratio of the diameter to its circumference and various methods used in its computation which are found in Sanskrit and Persian texts.

Āryabhaṭa (b. 476 A.D.) is the first mathematician in India who gives the expression for a much accurate approximate value of this ratio. He states the rule as :

> "Hundred plus four multiplied by eight and (combined with) sixty-two thousand is the approximate circumference of circle of diameter twenty thousand".[3]

The first use of the symbol π for the ratio of circumference to diameter was by William Jones in 1706, though in 1647 William Oughtred had used δ/π to represent the ratio diameter/periphery.[4]

That is :

$$\pi_a = \frac{(100+4)\times 8+62{,}000}{20{,}000}$$

$$= \frac{62832}{20000} \simeq 3.1416.$$

which is a better approximation and is in excess of

$$\pi_a - \pi \simeq 3.1416 - 3.141592$$
$$> 8.10^{-5}.$$

Cajori remarks that Āryabhaṭa himself never utilised this value, nor did any other Indian mathematician before the twelfth century.[5] But this appears in the works of Lalla (*c.* 749 A.D.), Bhaṭṭotpala (960 A.D.) and Śrīdhara (850 or 950 A.D.).

Brahmagupta (628 A.D.) states :

"the circumference is $3\frac{1}{7}$ times the diameter".[6]

According to al-Bīrūnī, Brahmagupta finds this number by a method peculiar to him. He says "as the root of 10 is nearly $3\frac{1}{7}$, the relation between the diameter and its circumference is like the relation between one and the root of 10".[7]

Archemides computed this ratio as $\pi_{ar} = \frac{22}{7}$.

The same value is found for the first time in the work of Āryabhaṭa the younger,[8] who did not give a method of computation as Archemides had given, to which we will return later. He takes this value as *sūkṣma* (accurate) but it has an error of

$$\pi_{ar} - \pi = 22/7 - 3.141592$$
$$\simeq 3.142857 - 3.141592$$
$$> 10^{-3}$$

Comparing with Āryabhaṭa's value, we have

$$\pi < 3.1416 < 3.1428.$$

But Bhāskara (1150 A.D.) and the author of *Kriyākramakarī* remarked this value to be *sthūla* (gross). This indicates that these mathematicians were aware of the fact that this value was not exact.

The fourteenth century mathematician Nārāyaṇa found much closer approximation to this ratio. According to him[9] :

$$\pi_n = \frac{355}{113} \simeq 3.1415929$$

This approximation is the same as that given by a Chinese mathematician Tsu Ch'ung-Chin in the fifth century. This accurate value was not to be arrived at in Europe until the sixteenth century.[10]

However, these expressions and results do not give even a word of preliminary explanation about how such results were arrived at, and this style of representation continued even in early nineteenth century by some mathematicians. *e.g.* Śaṅkaravarman (1800—1838 A.D.) who gives the expression "if the diameter of a great circle measures one *parārdh* ($= 10^{18}$), its circumference will be 314159265358979324"[11] which gives the value of $\pi_s = 3.14159265358979324$ which is correct up to 17 places of decimals. But of course from the later part of medieval period some other mathematicians changed this traditional method and gave a procedure for computation of this ratio. So Jyeṣthadeva (1475 and 1575 A.D.) in his *Yuktibhāṣā*[12] gives a geometrical method and finally arrived at an infinite series

$$\pi = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \quad \ldots\ldots\ldots\ldots\right)$$

which is the same as the series obtained by Gregory in 1668.

The method applied by Archemides was to compute the value from the perimeter of a regular inscribed and circumscribed polygon of sides 3.2^n, where $n = 1, 2, 3 \ldots\ldots$ So from the perimeters of given regular inscribed and circumscribed polygons, we may obtain the perimeters of the regular inscribed and circumscribed polygon having twice the number of sides. By successive application of this process, starting with the regular inscribed and circumscribed six-sided polygon, we can compute the perimeters of the regular inscribed and circumscribed polygons of 12, 24, 48, 96, ... sides. In this process, the main idea is that as the number of sides of regular inscribed or circumscribed polygons increases the perimeters of the polygon tend to the circumference of the circle and hence the value of π approaches accuracy.

This method was also adopted by some Indian mathematicians. According to Cajori, a commentator on *Līlāvatī*[13], these values (Āryabhaṭa's and Bhāskara's) were calculated by beginning with a regular inscribed hexagon, and applying repeatedly the formula

$$AD = \sqrt{2 - \sqrt{4 - AB^2}}$$

wherein AB is the side of the given polygon, and AD is that of the one with double the number of sides. In this way we obtained the perimeters of the inscribed polygons of 12, 24, 48, 96, 192, 384 sides. Taking the radius $= 100$, the perimeters of the last one gives for $r = 1$, we have the following values :

No. of Sides	Magnitude of the Side	Value of π
12	0.5176380907	3.1058288544
24	0.2610523855	3.132628626
48	0.1308062613	3.139350271
96	0.06563816928	3.141032125
192	0.03272347781	3.14145387
384	0.01636230423	3.141562412
768	0.008181259071	3.141603483
1536	0.004090721207	3.141673887
3072	0.002045482828	3.141861624
6144	0.001023230179	3.14336311

It appears from these results that the perimeter of the 768 sided polygon and not of the 384 sided , gives the value which Āryabhaṭa used for π.

The polygon of 384 sides gives good approximate value correct to four decimal places. But the formula fails in giving further more accurate value because at each step of the doubling operation the value computed is in excess of what should have come in increasing order.

An early nineteenth century mathematician and astronomer G͟hulām Ḥusain Jaunpūrī (1790-1862 A.D.), whose method of computation for this ratio is basically classical, adopted an excellent new technique. He takes directly an arc of a very small part of circumference *e.g.* one minute and describes equilateral and equiangular polygon whose sides are 21600 *i.e.* the number of minutes in the whole of circumference and obtains accurate value upto so many decimal places. Tytler remarks that "all the other Mohammaden mathematicians whom I had ever seen, contented themselves with the coarse approximation of 7 to 22, but it will here be seen that the author carries it on to seven places of decimals".[14]

In giving detailed method of this computation of the chord of 1′ G͟hulām Ḥusain states[15] :

"Since the chords of 20° and 12° are known, by exposition II[16] the chord of their difference which is 8° is also known. By exposition III[17], the chords of bisections of 8° *i.e.* of 4°, 2°, 1°, and 30′, 15′ are obtained. By previous exposition[18] the chord of 5′ is also calculated and that is $0°5'14''9'''32^{iv}30^{v}$, (where $1' = 60^{-1}$, $1'' = 60^{-2}$

and so on); and the chord of its 1/4 which is $1'15''$ is also known which is $0^{\circ}1'18''32'''23'^{v}7^{v}30^{v'}$.

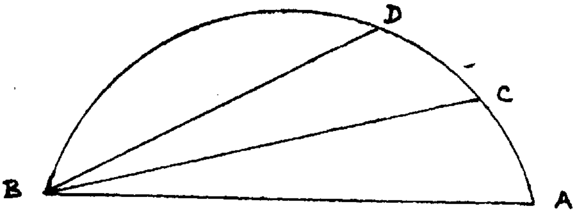

Fig. 1

"We suppose that the arc *AB* is $1'15''$ and the arc *CB* is $1'$ and the chord *AB* is known and the chord *CB* is not known. By proposition 57 of 4 of Chapter One[19] the ratio of the chord *AB* to chord *CB* is less than the ratio of their arcs *i.e.* the ratio of 5 to 4. Hence the chord of one minute will be greater than 4/5 chord of $1'15''$ which is $0^{\circ}1'2''49'''54'^{v}30^{v}$.

"Then we suppose that the arc *DB* is $50''$ which is 2/3 of the arc *AB* ($1'15''$) whose chord is known. Hence according to the earlier rules its chord comes out to be $52'21''35'''25'^{v}3^{v}$. Then we say that the ratio of *CB*, the chord of one minute to *DB*, the chord of $50''$ will be less than the ratio of one minute to $50''$, *i.e.* the ratio of 6 to 5, and the number whose ratio to the chord of $50''$ is as 6 to 5 is also $0^{\circ}1'2''49'''54'^{v}30^{v}$. Hence the chord of one minute will be smaller than $0^{\circ}1'2''49'''54'^{v}30^{v}$ and the chord of $1'$ some time exceeds the number correct to the fifths and some time gets lesser than the same number. Thus it is clear that the difference of the chord of $1'$ from this number upto the fifths is not really felt. Hence the chord of $1'$ will be $0^{\circ}1'2''49'''54'^{v}30^{v}$."

Ghulām Ḥusain derived this ratio as follows[20]:

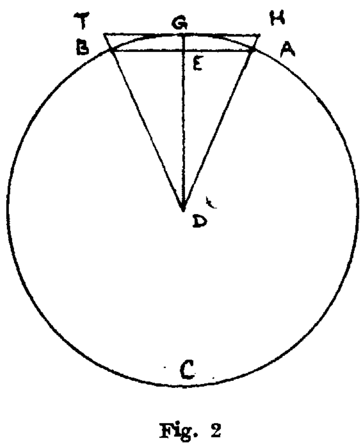

Fig. 2

"We suppose an arc AB, a very small part of the circumference for example, one *daqīga* (minute), and the magnitude of the chord $AB = 1'2''49'''54'^{v}30^{v}$, and the centre of the circle is the point D. We join DA, DB, the two radius and from the centre D draw a perpendicular DE to AB. By proposition 3 of 3 of Chapter One, this perpendicular will bisect the said chord at the point E. The perpendicular is produced on the side of E to the point G, which is on the circumference. We draw from G the perpendicular GT to DG till it meets DA and DB produced, at the points H and T. Then we say that when the known square of AE which is

$$AE^2 = \ldots\ldots\ldots\ldots\ldots\ldots 16''' \; 26'^{v} \; 57^{v} \; 32^{v'} \; 12^{v''} \; 37^{v'''} \; 33'^{x} \; 45^{x},$$

we get

$$\begin{aligned} DE^2 &= DA^2 - AE^2 \\ &= 59 \text{ elevation } 59°59'59''45'''33'^{v}2^{v}37^{v'}47^{v''} \\ &\qquad 22^{v'''}26'^{x}15^{x} \end{aligned}$$

or

$$DE = 59°59'59''59'''51'^{v}46^{v}$$

and due to similarity of the two triangles DEA and DGH, the ratio of the known side DE to DG will be as the ratio of the known side AE to unknown side GH that is,

$$\frac{DE}{DG} = \frac{AE}{GH}$$

Therefore

$$\begin{aligned} GH &= \frac{AE.DG}{DE} \\ &= 31'24''57'''15'^{v}4^{v}. \end{aligned}$$

Therefore

$$\begin{aligned} HT &= 2\,GH \\ &= 1°\,2'\,49''\,54'''\,30'^{v}\,8^{v}. \end{aligned}$$

Now since AB is the chord of one minute, there is no doubt that it is the side of an equilateral and equiangular figure which is described in the circle and the number of whose sides is 21600 and in fact it is six duplication (6×60^2). If we multiply AB by six duplications, it will be the length of all sides of the given figure, *i.e.*

$$\Sigma\, AB = 376°59'27''$$

and the circumference of the circle is greater than this quantity, by proposition 50 of 4, Chapter One.[21]

"Again, HT is the side of the same figure which is described about the circle, and if we multiply HT into the same six duplication the length of all the sides of the figure about the circle will be obtained.

$$i.e.\ \Sigma\, HT = 367°59'27''0'''48'^{v}$$

and the circumference of the circle is smaller than this, by proposition 51 of 4, Chapter One.[22]

"Hence the circumference of the circle is as it were the mean number of these two. So if we take half the difference which is $24'^{v}$ and either add it to the said smaller number or subtract it from the greater, in both the cases the amount of the circumference of the circle is obtained

$$\begin{aligned} \text{i.e. circumference} &= 1/2\,(\Sigma\, AB + \Sigma\, HT) \\ &= 367°59'27''0'''24'^{v}. \end{aligned}$$

Now, divide the quantity of the circumference by the diameter which is two elevation, that is 120. We get the ratio of diameter to its circumference, *i.e.*

$$\pi = 3°8'29''43'''30'^{v}12^{v}$$

$$\text{or} = 3+8.60^{-1}+29.60^{-2}+43.60^{-3}+30.60^{-4}+12.60^{-5}$$

$$= 3.141590."$$

This value is correct upto five decimal places.

As we know, the perimeter of inscribed or circumscribed polygon can not attain the circumference of the circle in any case by increasing the number of sides of the polygon, but the difference can be minimised by taking the mean of the perimeters of circumscribed and inscribed polygons with greater number of sides as G͟hulām Ḥusain did. Of course, in this method the accuracy of this proportion depends on the accuracy of the chord length of the arc measured which subtends an angle of 1′ on the centre of the circle.

Realising the importance of his result G͟hulām Ḥusain states :[23]

"the value which is common among surveyors is that the proportion of the diameter to the circumference is as the proportion of 7 to 22. This proportion is less than the accurate proportion which has been stated, for if we reduce the proportion of 7 to 22 to decimal fractions, it is as the proportion of unity to this number 3.1428571 and this is greater than the first by the fraction 0.00012668, but as this excess is approximately one part out of a thousand parts, so in the measurement of small circles the difference is not perceptible and this is the proportion generally employed".

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References and Notes

[1]*Alberuni's India*, Tr. by E. C. Sachau, S. Chand and Co., 1964, p. 165.

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[3]Gupta, R. C., Āryabhaṭa I's Value of π, *The Mathematical Education*, **VII**, (1), 17, 1973.

[4]Pottage, J., The Vitruvian Value of π, *Isis*, **59**, 190, 1968.

[5]Cajori, F., *A History of Mathematics*, Chelsea Publishing Co., New York, 1980, p. 87.

[6]Datta, B. B. and Singh, A. N., Hindu Geometry, revised by K. S. Shukla, *Indian Journal of History of Science*, **15**, (2), 154, 1980.

[7]See ref. 1, p. 168.

[8]Datta, B. B. and Singh, A. N., *I.J.H.S.*, p. 154.

[9]Datta, B. B. and Singh, A. N., *I.J.H.S.*, p. 155.

[10]Swetz, F., The Evolution of Mathematics in Ancient China, *Mathematics Magazine*, **52**, (1), 15, 1979.

[11]Datta, B. B. and Singh, A. N., *I.J.H.S.*, p. 156.

[12]Sarasvati Amma, T. A., *Geometry in Ancient and Medieval India*, Motilal Banarasidass, Delhi, 1978, p. 12.

[13]Perhaps this commentator is Gaṇeśa. As Sarasvati Amma states (p. 156) "Gaṇeśa suggests that the side of the 384-sided polygon inscribed on a circle of diameter 100 was calculated by repeated application of the formula

$$S_{2n} = \sqrt{\frac{1}{4} S_n^2 + \left(r - \frac{1}{4}\sqrt{4r^2 - S_n^2}\right)^2}$$

where S_n and S_{2n} are the sides of the polygon of n sides and $2n$ sides respectively inscribed in the circle". According to the present author's verification this formula is equivalent to that attributed to the commentator.

[14]Tytler, J., Analysis and Specimens of a Persian Work on Mathematics and Astronomy, *Journal of Royal Asiatic Society of Great Britain and Ireland*, **IV**, 258, 1837.

[15]G͟hulām Ḥusain Jaunpūrī, *Jāme-i-Bahādur Khānī*, Calcutta, 1835, pp. 344–345.

[16]*Ibid.*, Deals with the chord of the difference of two arcs whose chords are known. pp. 339–340.

[17]*Ibid.*, Deals with the chord of the half arc whose chord is known. pp. 340–341.

[18]*Ibid.*, Deals with the trisection of an arc. pp. 342–344.

[19]*Ibid.*, In unequal arcs, which are not greater than a semicircle, the ratio of the chord of the bigger arc to that of the smaller arc will be less than the ratio of the bigger arc to the smaller arc. p. 66.

[20]*Ibid.*, p. 426.

[21]*Ibid.*, In every rectilineal surface which is inscribed in a circle the sum of its sides will be less than the circumference of the circle.

[22]*Ibid.*, In every figure which is circumscribed on a circle, the sum of its sides will be greater than the circumference of the circle. p. 63.

[23]*Ibid.*, p. 427.