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NON-EUCLIDEAN GEOMETRY FROM EARLY TIMES TO BELTRAMI

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This is a brief survey of the development of Non-euclidean geometry from early times to the end of nineteenth century. It highlights the attempts of several mathematicians who tried in vain to give a solid proof of Euclid's parallel postulate. Then it describes that how by separating the parallel postulate from the other postulates, the nineteenth century mathematicians Gauss, Janos Bolyai, Lobachevski and Riemann developed a logically consistent system of Non-euclidean geometry. The paper ends with the contributions made by Klien and Beltrami highlighting the philosophical implications of this system of geometry.

'No experiment will ever come in conflict with the postulates of Euclid ; on the other hand, noexperiment will ever contradict the postulates of Lobachevski.'

— Earnst Cassirer[1]

I

It is difficult to say *what is geometry*, but to the ancient minds it was a physical science used as a significant tool for construction. They developed suitable empirical formulae to define areas and volumes of various types of surfaces and solids. Today the subject is so vastly developed that it has become a special branch of mathematics. One of the prominent ancient minds who took interest in geometry was Pythagoras (6th century B.C.) of Samos, known for his contributions to philosophy and mathematics. He and his disciples had used 'form' in the 'service of numbers' and solved many numerical problems by geometrical methods. The well known Pythagoras theorem was vital later to development of analytical methods in geometry. They developed the concept of 'proof of a theorem'. The work on geometry continued for nearly two hundred years after Pythagoras but the world was yet to see a book of reference on geometry. Then, during the third century B.C. Euclid, a famous scholar of Alexandria, brought out his famous *Elements* consisting of thirteen books. The fifth, seventh, and the tenth books are devoted to the theory of proportions and arithmetical sets in geometrical form, and the remaining books deal with geometry proper. While writing his book Euclid had difficulties in successfully defining exactly what is meant by a point, straight line, and equality of two distances in space. Also he did not consider the concept of 'betweenness' which was found to be a serious gap by the later geometers. He gave definitions and postulates to derive the geometrical properties of the plane. It is the famous fifth postulate which raised controversies and made scholars to develop non-Euclideangeometry. The fifth postulate is given as :

> 'that if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines if extended indefinitely meet on the same side on which the angles are less than two right angles.'

This is also known as Euclid's parallel postulate. This is explained as :

> 'that only one straight line can be drawn parallel to a straight line through a point not on this straight line.'

While proving nearly twentyeight of his propositions Euclid never used this postulate. It appears that Euclid himself considered this postulate inferior to the other postulates. Some of the defects in Euclid's *Elements* was noticed by many scholars. The famous ancient scientist Archimedes of Syracuse extended the list of Euclid's postulates and almost completed Euclid's exposition of theory of length, areas, and volumes. Euclid had established ratios, *etc.* regarding areas, volumes, *etc.*, whereas Archimedes obtained expressions which could be used to calculate the various above mentioned measures. Some of the postulates of Archimedes were later found unacceptable. Euclidean geometry was later fully explained by Apollonius who also made valuable contributions to astronomy as also geometry of conic sections. Proclus (500 A.D.) in his commentary on Euclid sounded a warning about the parallel postulate by suggesting not to make undue use of the intuitive evidences. Arabs succeeding the Greeks as leaders in the development of geometry also investigated the fifth postulate. A special mention should be made of Nasīr-al-Dīn[2] at Ṭusī (1201-1274), a founder of an astronomical observatory in Persia (now Iran), for his original idea of clearly putting in the forefront the theorem on the sums of the angles of a triangle and exhaustive discussions. Using a right angled traingle he proves that the sum of the angles is equal to two right angles and then extends his arguments by constructing a diagram to consider the other cases and give a rigorous proof of the fifth postulate. Ultimately he did not succeed. After him many scholars tried in vain. This exercise went on till the end of the seventeenth cntury. Only during the eighteenth century and after, attempts were made seriously to question the validity of Euclid's fifth postulate and the consequent development of non-Euclidean geometry which can be divided into two phases ;

i. forerunners of non-Euclidean geometry
ii. founders of non-Euclidean geometry

II

One of the prominent forerunners of non-Euclidean geometry was the jesuit Italian mathematician Girolamo Saccheri[3] (1667-1733) who tried to free Euclid of every flaw as the title of his book, which was published during the year of his death, suggests. He tried to prove the fifth postulate by assuming its opposite and then contradict it by a logical mathematical proof—*reductio ad absurdum.* He considered a quadrilateral *ABCD* where the angles *A* and *B* are right angles and the sides *AD* and *BC* are equal, and then $\angle C$ is equal to $\angle D$. He considered three hypotheses ;

i. hypothesis of right angle *i.e.* angles *C* and *D* are right angles ;
ii. hypothesis of obtuse angle *i.e.* angles *C* and *D* are obtuse ; and
iii. hypothesis of acute angle *i.e.* angles *C* and *D* are acute.

Since the right angle hypothesis is equivalent to Euclid's fifth postulate, Saccheri had to show that the other two hypotheses are unacceptable. By precise arguments, Saccheri showed that the obtuse angle hypothesis leads to contradiction. Then assuming the hypothesis of acute angles to hold good, he arrived at important results but at the end arrived at two contradictions. In trying to develop a final proof, his approach involved a lot of complications and did not lead to any final result.

Then we come across the contributions of the Swiss naturalist and mathematician John H. Lambert[4] (1728-1777) who also tried to give a proof of the validity of Euclid's fifth postulate. It appears that, as mentioned by Klugel[5] (1739-1812), Lambert was familiar with the worksof Saccheri. Lambert divides the problem into three parts : the first two dealing directly with the Euclid's fifth postulate while the third part contains an investigation. He considers a quadrilateral with three right angles and the three hypotheses are made as to the nature of the fourth angle *i.e.* it is (i.) a right angle, (ii.) an obtuse angle, and (iii.) an acute angle. Lambert's approach was similar to Saccheri's and he could not decide the issue regarding the acute angle hypothesis. But his investigations contain many interesting parts, the one being a close resemblance to spherical geometry which could hold good if the second hypothesis is valid and a remark that spherical geometry is independent of the parallel postulate. He brought out the concept of 'defect of a triangle' which is defined as the difference between two right angles and the sum of the angles of the traingle. This defect is proportional to the area of a traingle.

After Lambert the famous French mathematician Adrian Marie Legendre[6] (1752-1833) also tried to give a solid proof of the fifth postulate. He examined the problem by three mutually exclusive possibilities :

i. the sum of the angles of a traingle is greater than two right angles ;
ii. the sum of the angles of a triangle is equal to two right angles ; and
iii. the sum of the angles of a traingle is less than two right angles.

By subtle arguments Legendre shows that the first possibility leads to a contradiction. As regards the third possibility he tried for a long time to contradict it and did not succeed. While tackling this part of the problem he inadvertently used an equivalent form of Euclid's parallel postulate. Also his arguments had close connection with those of Saccheri and Lambert.

The last prominent forerunner of non-Euclidean geometry was the Hungarian mathematician Farakas Bolyai[7] (1775-1856), a contemporary and close friend of Gauss. In 1804 Bolyai sent Gauss a paper titled 'Theoria Parallelarium' which was an attempt at a proof of the equidistant straight line. Gauss, usually a careless man, went through the proof in detail and showed that the proof was fallacious. Farakas Bolyai did not succeed to rectify the mistake but he could point out a postulate to which he reduced Euclid's postulate. The postulate is 'four points not in a plane always lie on a sphere or this implied that a circle can always be drawn through three points not in a straight line.' This also suggested that the existence of such a circle should be established as a preliminary investigation of parallels. Taking a hint from this, F.L. Wachter[8] (1792-1817), a student of Gauss made several attempts to prove the fifth postulate. His letter to Gauss written in 1816 deserves a special mention for after a conversation with Gauss they both thought of *Anti-Euclidean Geometry.* Wachter later thought that even in case of the fifth postulate being false there would be a geometry on the surface identical with that of ordinary plane. This statement is of special importance as it foresees the development of non-Euclidean geometry and also the theory of acute angle discussed by Saccheri.

III

By the beginning of the nineteenth century analytical geometry developed by Rene Descartes was extended to three dimensions by the French mathematician Alexis Clairaut and others. Also Poncelet, Charles Jacob Steiner, and Von Staudt developed projective geometry where the system was free from the measuring process. Their efforts were encouraging and the mathematical world could see the synthesis of the different systems of geometry therby resulting in better methods of tackling problems of geometry. The famous German mathematician Karl Fredrich Gauss[9] (1777-1855), while tackling problems of land surveying and measurements, brought out a memoir in 1827 titled *General Investgations of Curved Surfaces.* In this paper he developed what is now well known as differential geometry where extensive use of differential calculus is made to arrive at suitable formulae for measuring curvature of surfaces. He also went one step further and tried to separate out the parallel postulate from the other postulates and develop a consistent geometrical system. He was afraid to make his results public for he thought that the academic world would boo him out !

Working on similiar lines Nicholas Lobachevski[10] (1792-1856) a professor of mathematics at Kazan University in Russia brought out a paper in 1828 and four years later, in 1832, Janos Bolyai[11] (1802-1860) son of Farakas Bolyai also brought out a paper which were foundations of non-Euclidean geometry. Janos Bolyai sent his paper to Gauss for comments. Gauss, known for his carelessness, never bothered to comment at all for he was afraid of publishing his own paper on the topic. Lobachevski had shown in his paper that by assuming more than one parallel can be drawn to a line from a point not on the line, he could develop a consistent geometry. He called his geometry *Imaginary geometry*. In this geometry the Lobachevskian plane was defined as the imaginary radius sphere in the pseudo-Euclidean space with identified diametrically opposite points. The diamteric sections of the sphere similiar to the great circles of the ordinary sphere play the role of straight lines on a Lobachevskian plane. It was verified that the tangent planes to this sphere were Euclidean and so the geometry within small areas of such spheres as in the cases of within the small area of ordinary sphere differ slightly from Euclidean geometry. The Lobachevskian plane contains three classes of extrarodinary curves known as the circle, the equidistants, and the oricycle which are defined as :

i. a circle is as in ordinary plane is the locus of points of the plane lying at the same distance from a specified point. (As in ordinary plane, circles in Lobachevskian plane can also be defined as curves such that in each of their points interesect at right angles a bundle of straight lines intersecting in a common point.) ;

ii. an equidistant curve is a curve separated by equal distances from all points of a given straight line called the base of the equidistant curve. (In Lobachevskian plane such loci are not pairs of straight lines but curves consisting of two branches.) ;

iii. oricycle or herocycle is defined as a curve which, in each of its points, intersect at right angles the straight line parallel to each other. (Oricycle means the limiting circle.)

Poincare using the method of inversion gave interpretations of Lobachevskian plane.

Lobachevskian geometry known as *Hyperbolic Geometry* widened the frame work of the Euclidean system. His geometry found direct applications in the theory of definite integrals and other areas of mathematics. He showed that in his geometry the sum of the angles of a triangle is less than two right angles where the sides of the triangle are arcs of circles which if produced would cut orthogonally the fundamental plane. Also the sum of the angles of a triangle is not constant and it diminishes as the area of the triangle increases. Also no triangles having unequal areas are ever similar and the ratio of the circumference of a circle to its diameter is not constant for all circles and is always greater than and increases with areas.

Following the trio, George Bernard Riemann[12] (1826-1866), a brilliant student and later a successor of Gauss at Gottingen read a paper titled 'on the hypotheses that form the foundations of geometry' before an audience in 1854. This paper shook the mathematicians' world. Riemann stressed the idea that a system of geometry which is self-consistent could be constructed without assuming the truth of Euclid's parallel axiom. He totally rejected this postulate. He pointed out that in all previous investigations there was the assumption that the straight line was of infinite length which was natural and obvious to the common eye. He questioned—what would happen if this assumption is given up and allow the straight line to return to itself as in the case of a great circle of a sphere ? Here one faces the difference between infinity and unboundedness of space which can be easily comprehended in a two dimensional space. We observe that both the surfaces of sphere and ordinary plane are both unbounded but the sphere is finite whereas the plane is infinite. Taking the sphere as a basis and considering the two extreme points of a diameter of a sphere known as antipodal points any number of great circles can be drawn, *i.e.*, if A and B are the two antipodal points any number of great circles can be drawn whereas in Lobachevskian geometry and Euclidean geometry only one line can be drawn. Also the two great circles enclose an area whereas in the Euclidean geometry the two straight lines does not cover an area. In the case of points on the sphere the concept of betweenness does not hold good.

Carrying out his investigations further Riemann generalised the differential geometry developed by Gauss by stating that notion of geodesics (great circle arc on a sphere) and curvature could be employed in connection with spaces having any number of dimensions. He brought out the relation between the geometry of space and curvature. If the curvature of space is positive and constant, the space is Riemannian type of non-Euclidean geometry and if it is negative and constant then it is Lobachevskian type of non-Euclidean geometry and if it is zero then it is Euclidean. He showed that the Pythagorean formula is a highly specialised one *i.e.* it holds good if one restricts himself to infinitesmal volumes ând so then Euclidean geometry holds good. So far the two dimensional Riemannian geometry has been explained by using a sphere which is of three dimensions.

The famous Norwegian mathematician Sophus Lie[13] (1844-1899), explained the various types of geometry by the theory of transformation groups, and how they are inter-related. Also Cayley had shown that how metrical geometry can be viewed also as projective and explained the power of projective geometry in tackling problems of Euclidean geometry. Felix Kliên[14] (1849-1925), who was already working on these problems, succeeded in showing that the differences between the three types of geometry may be viewed as a consequence of the fact that each of these systems employs a different definition of congruence or measure of distance and angle. Also in the case of Riemannian geometry the sum of the three angles of a triangle is greater than two right angles.

Klien compared the three systems of geometry to a solution of an ordinary quadratic equation—for it can have (i.) two identical real roots, (ii.) two different real roots, or (iii.) no real roots, which corresponds respectively to Euclidean geometry where only one parallel can be drawn to a line through a point, Lobachevskian geometry where more than one parallel can be drawn, or Riemannian geometry where no parallel can be drawn.

The interpretations of non-Euclidean geometry was left to Eugino Beltrami[15] (1835-1900) who brought out an important paper on this topic. He showed that under certain conditions the plane Euclidean geometry can be considered as an intrinsic geometry of some surface. He tackled two main questions :

i. to find a surface for each point of which there is a neighbourhood which is isometric to a certain region in a Lobachevskian plane, and
ii. to find a surface on which an isometry can be defined that maps the surface into an entire Lobachevskian plane.

Beltrami solved the first question. An affirmative answer to the second question would have implied logical consistency of the two dimensional non-Euclidean geometrical system. He was unsuccessful in giving a solution. While investigating the consistency of Lobachevskian geometry Beltrami made use of a surface known as the pseudo-sphere. This surface is generated by rotating the curve 'Tracitrix' about its asymptote. The pseudo-sphere has a negative constant curvature. Beltrami proved a theorem which states—'that in the neighbourhood of every point of a pseudosphere Lobachevskian geometry holds good.'

Clifford[12] (1845-1879), who had translated Riemann's paper discussed Euclid's parallel straight lines by excluding the property that they are coplanar but included the other two properties, *i.e.*, (i.) they have no common point and (ii.) they are equidistant. He explained the non-planar parallels for an elliptic space and their remarkable properties. Then he explained the properties of surfaces (Clifford surfaces) generated from the above system of parallel lines. Then Klien tackled the issue, *i.e.*, the determination of all the two dimensional manifolds of constant curvature which are regular everywhere. As regards Euclidean space the answer was negative. This was extended to investigate Clifford-Klien forms for positive or negative values of curvature.

These developments in non-Euclidean geometry had philosophical implications. This made the mathematicians reject Kant's concept that the axioms of geometry are synthetic *a priori*. In fact Riemannian geometry had a big effect on later developments in physics and cosmology as Hermann Weyl[16] (1885-1955) says "Riemann left the development of his ideas in the hands of some subsequent scientist whose genius as a physicist would rise to equal flights with his own as a mathematician. After a lapse of seventy years the mission has been fulfilled by Einstein."

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Remarks : Particularly the books by R. Bonola, N. Efimov, and Felix Klien's Geometry cover in detail (all the above books are in English translation) the development of the various types of geometry, their inter-relations, *etc.*