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# SUNDARARAJA'S IMPROVEMENTS OF VEDIC CIRCLE-SQUARE CONVERSIONS

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If a square is converted into a circle of equal area, the traditional vedic method (as found in major *Śulba Sūtras*) implies

$$d = (2 + \sqrt{2})\, s/3 \qquad \text{... (A)}$$

where s is the side of the square and d is the diameter of the circle. For the reverse process of converting a given circle into an equal square, a popular vedic rule was based on

$$s = 13d/15 \qquad \text{... (B)}$$

Some other rules were also known, e.g.

$$d = 9s/8 \text{ and } s = 8d/9 \qquad \text{... (C, D)}$$

Sundararāja (*c.* 1500 AD or earlier?), a commentator of *Āpastamba Śulba Sūtra*, suggested improvements in these rules to get better results. His method amounts to using a corrcting multiplicative factor on the right-hand side expression of each rule. His numerical values of the correcting factor are: 117/118, 136/133, 331/330, 331/332 respectively for (A), (B), (C) and (D).

Significantly, it is shown in the paper that all these corrections were probably obtained by utilizing Āryabhaṭa I's famous value of $\pi$ (= 62832/20000) and then applying the ancient approximation

$$\sqrt{(a^2 + x)} = a + x/(2a + 1)$$

## 1. INTRODUCTION

In spite of several attempts, the Indus script has not been deciphered successfully. So, we are not in a position to say definitely about many matters related to the achievements of the Indus Valley Civilization (*c.* third millennium BC), which is believed to be older than the Vedic Civilization in India. Thus, India's most ancient written works may be taken to be the *Vedas*, which are four in number, namely *Ṛg-Veda* (the oldest of them), *Yajur-Veda*, *Sāma-Veda* and the *Atharva-Veda*. Also, there are different branches and schools which are represented by the various *Saṁhitās* or recensions of the *Vedas*.

To assist the proper study of the *Vedas*, there are six *Vedāṅgas* ("limbs or parts of the *Veda*"), namely, *Śikṣā* (phonetics), *Kalpa* (ritualistics), *Vyākaraṇa* (grammar), *Nirukta* (etymology), *Chandas* (prosody and metrics), and *Jyotiṣa* (astronomy, including mathematics and astrology). The *Kalpa* deals with the rules and methods for performing vedic rituals, sacrifices and ceremonies, and is divided into three categories which are called *Śrauta*, *Gṛhya* and *Dharma*. The *Śrauta* texts are generally called *Śrauta Sūtras*, because they are written in the *sūtra* or aphoristic style. They are more specifically concerned with the Vedic sacrificial ritual and allied ecclesiastical matters.

The *Śrauta Sūtras*, especially those belonging to the various *Saṁhitās* of the *Yajur-Veda*, often include tracts which give rules concerning the measurements and construction of *Vedic* (sacrificial grounds), *citis* (mounds or altars), and *agnis* (fire-places). Such tracts are also found as separate works and are called *Śulba Sūtras* or *Śulbas* simply. These tracts are the oldest geometrical treatises which represent, in coded form, the much older and traditional Indian mathematics developed for construction of Vedic altars of various types and forms. The word *śulba* is derived from the root *śulb* (or *śulv*) meaning "to mete out" or "to measure".

By now the names of about a dozen *Śulba Sūtras* are known. These are: *Baudhāyana, Āpastamba, Kātyāyana, Mānava, Satyāṣāḍha, Maitrāyaṇīya, Varāha, Vādhūla* (or *Bādhūla*), *Maśaka, Hiraṇyakeśī*, and *Laugākṣī*. They are variously dated and exact times of their composition or compilation are controversial. *Baudhāyana Śulba Sūtra* (henceforth abbreviated as *BSS* ) is the oldest of them and is generally placed between 800 BC and 500 BC. The *Āpastamba Śulba Sūtra (= ASS), Kātyāyana (or Kātīya) Śulba Sūtra* (=*KSS*), and *Mānava Śulba Sūtra* (=*MSS*) are other important ancient works of the class. The text of *Satyāsāḍha Śulba Sūtra* is said to be identical with that of *ASS*[1] (which is the second oldest work of the class), while *Maitrāyaṇīya Śulba Sūtra* is said to the another but different version of *MSS*[2]. More than a dozen commentaries are extant on these *Śulbas*. *BSS* was commented by Veṅkateśvara (or Vyaṅkateśvara) and Dvārakānatha, *ASS* by Kapardi, Karavinda, Gopāla and Sundararāja, *KSS* by Karka, Rāma Vājapeyin, Mahīdhara (or Mahīdāsa), and Gaṅgādhara, and *MSS* by Śivadāsa, whose younger brother Śaṅkara commented on the *Maitrāyaṇīya Śulba Sūtra*.

## 2. Vedic Circle-Square Conversions

We now know that exact construction by ruler and compass for converting a square into a circle of equal area or vice versa is impossible, because $\pi$ is a transcendental number. Hence, all ancient attempts at circling a square or squaring a circle gave only approximate methods whether this nonexactness was mentioned or not.

In Fig.1 a given square ABCD of side s is shown along with its circum-circle whose centre is at O, and whose radius is given by

$$OA = s/\sqrt{2} = \sqrt{2}\, s/2$$

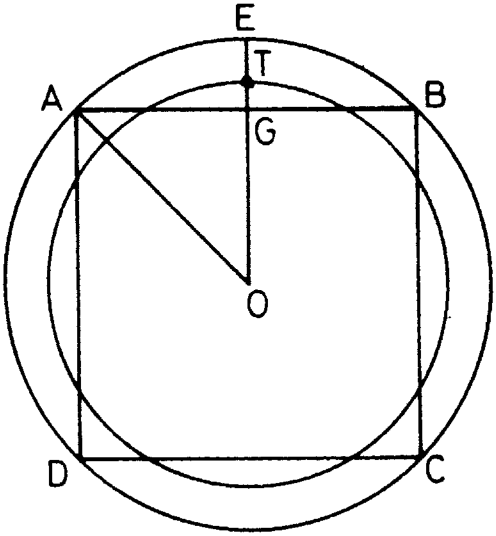


Fig. 1

Let G be the middle point of AB and let OG produced meet the circumcircle at E (in ancient diagrams the east direction was taken upwards). If r is the radius of the circle whose area is exactly equal to that of the given square, we must have

$$\pi r^2 = s^2 \qquad \dots (2)$$

or

$$r = s/\sqrt{\pi} \qquad \dots (3)$$

$$= 0.5642\ s, \text{ nearly} \qquad \dots (4)$$

Also

$$OE = OA = \sqrt{2}\ s/2 \qquad \dots (5)$$

$$= 0.7071\ s, \text{ nearly} \qquad \dots (6)$$

Cleary, the point T, where OT = r, lies between G and E, which is otherwise obvious by considering the inscribed and circumscribed circles of the given square. As mentioned above, it is impossible to locate the truly exact position of T by ruler and compass construction. But several attempts were made to locate T approximately by simple considerations.

The Indian traditional vedic method, which is found in all the four main *Śulbas*, namely, *BSS*, *ASS*, *KSS* and *MSS*, is to take[3]

$$OT = OG + GE/3 \qquad \dots (7)$$

$$= s/2 + (\sqrt{2} - 1)\ s/6 \qquad \dots (8)$$

$$= (2 + \sqrt{2})\, s/6 \qquad \ldots (9)$$

$$= 0.569\ s, \text{ nearly} \qquad \ldots (10)$$

An equivalent rule found in *MSS*, 11.9-10 (pp. 66 and 135), has been given a better interpretation recently and implies a slightly different construction[4]. This is shown in Fig.2 in which we have

$$CE = CA = \sqrt{2}\, s$$

The diameter of the desired equivalent circle is then taken to be

$$CT = CB + BE/3$$

$$= s + (\sqrt{2} - 1)\, s/3$$

$$= (2 + \sqrt{2})\, s/3$$

$$= 2\ OT$$

Thus, the size of the circle obtained will be same whether we get it by constructing Fig. 1 or Fig. 2.

A few years ago, the present author had given a new interpretation of the second part of *MSS*, 11.15 (pp. 66 and 136), to suggest that the implied rule amounts to taking[5]

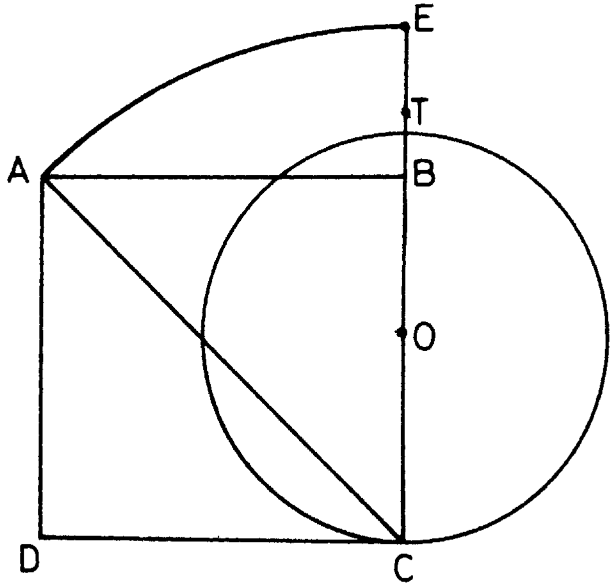


Fig. 2

$$OT = OE - OE/5 \qquad \ldots (11)$$

$$= (4/5)\ OE \qquad \ldots (12)$$

$$= 2\sqrt{2}\ s/5 \qquad \ldots (13)$$

$$= 0.5657\ s, \text{ nearly} \qquad \ldots (14)$$

The significance of this interpretation lies in the fact that it implies a new *Śulba* value of $\pi$ given by

$$\pi = s^2/(OT)^2 \qquad \ldots (15)$$

$$= 25/8 = 3.125 \qquad \ldots (16)$$

by using (13). This is in fact the best value of $\pi$ so far obtained by interpreting various rules from the texts of the *Śulba Sūtras*[6].

There is some evidence to show that the following rule was also known in India (Fig. 1):

$$OT = OG + OG/8 \qquad \ldots (17)$$

$$= 9\ s/16 \qquad \ldots (18)$$

$$= 0.5625\ s, \text{ nearly} \qquad \ldots (19)$$

Actually, relation (18) is the converse form of the rule

$$s = 8d/9 = 16r/9 \qquad \ldots (20)$$

which is mentioned below as Eq. (27).

For the other problem of converting a given circle (of diameter d) into an equivalent square, a simple construction is that which results from a new interpretation of *MSS*, 11.9-10 (p.66) recently given by Takao Hayashi[7]. According to this, an equilateral triangle VPQ is erected on a diameter PQ of the given circle (Fig. 3), and the *avalambaka* (altitude) VL is dropped from the vertex on PQ. Then the square VLMN constructed on VL as a side is the required figure. Here,

$$s^2 = (VL)^2 = d^2 - (d/2)^2 = 3d^2/4 \qquad \ldots (21)$$

Also, the aim of this construction was to make

$$\pi\ (d/2)^2 = s^2 = 3d^2/4 \qquad \ldots (22)$$

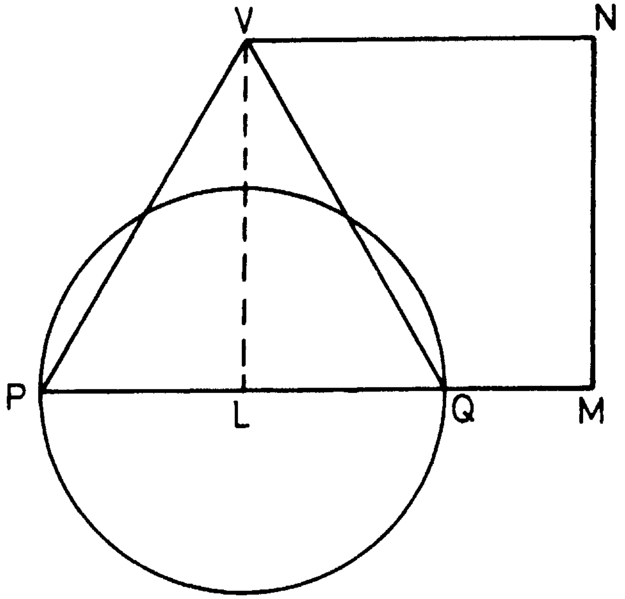


Fig. 3

implying the ancient simple approximation $\pi = 3$.

However, a more popular rule to solve the above problem was to divide the given diameter into 15 parts and take 13 of these to be equal to the side s of the desired square. That is,

$$s = 13d/15 \qquad \dots (23)$$

It is interesting to note that Sen and Bag[8] have given a simple rationale of this rule (23) by employing the value $\pi = 3$ (which is also implied in the above *MSS* rule) and the approximation

$$\sqrt{3} = 1 + (2/3) + (1/15) = 26/15 \qquad \dots (24)$$

Nevertheless, irrespective of any derivation, rule (23) implies the relation

$$(13\ d/15)^2 = \pi\ (d/2)^2 \qquad \dots (25)$$

which yields the value

$$\pi = 676/225 = 3.004444 \qquad \dots (26)$$

Verbal rules which give solution (23) are found in *BSS*, 2.11 (p. 19), *ASS*, 3.3 (p. 41), and *KSS*, 3.12 (p.56), but not in *MSS*, 11.13, as supposed (wrongly) by some scholars[9].

It is well known that for finding the area of a round field of diameter 9 *khet*, the Egyptian *Rhind Mathematical Papyrus* (*c.* 1650 BC), problem 50, uses the equivalent of the following rule[10]:

$$s = d - d/9 = 8d/9 \quad \ldots (27)$$

According to Khadilkar[11], this rule is found in *MSS*, but he does not quote the text or specific reference from this work. However, a published commentary of Mahīdhara (AD 1590) on *KSS* contains the following verse quoted from *Vārtika*[12]:

वृत्तव्यासं नवांशे वा परिहृत्याथ तां वदेत् ।
करणी चतुरस्रार्थमल्पमेवान्तरं भवेत् ॥

*Vṛttavyāsaṁ navāṁśe vā parihṛtyātha tāṁ vadet* ।
*Karaṇī caturasrārtham-alpamevāntaraṁ bhavet* ॥

'Leave out the ninth part of the diameter of a circle. That gives the side of the (equivalent) square with only minor difference'.

That is, we get rule (27). It may also be noted that Mahīdhara regards (23) to be approximate *(sthūla)* and (27) to be accurate *(sūkṣma)*; he puts the latter as:

$$s = d - 2d/18 \quad \ldots (28)$$

The more important question which arises in this connection is about the identification of the expository work *Vārtika* (from which the above verse is quoted) and of its author. I conjecture that it may be same as Rāma Vājapeyin's *Śulba Vārtika* (AD 1434), which is said to be a metrical gloss on *KSS* and on *Karka-Bhāṣya*[13].

It may be pointed out that verbal rules equivalent to (27) as well as its converse (18), were known to Sundararāja (a commentator of *ASS*), who in fact gave their corrected or improved forms (*see* the following Sections). We may also mention that *BSS*, 2.10 (p. 19), contains a special rule, which can be expressed as

$$s = d - 28\left(\frac{d}{8}\cdot\frac{1}{29}\right) - \left[\left(\frac{d}{8}\cdot\frac{1}{29}\right)\cdot\frac{1}{6} - \left(\frac{d}{8}\cdot\frac{1}{29}\cdot\frac{1}{6}\right)\cdot\frac{1}{8}\right] \quad (29)$$

This can be put in a more convenient and elegant form as

$$s = \frac{d}{1} - \frac{d}{8} + \frac{d}{8\,.\,29} - \frac{d}{8\,.\,29\,.\,6} + \frac{d}{8\,.\,29\,.\,6\,.\,8} \quad \ldots (30)$$

by using only unit fractions for representing s/d.

### 3. SUNDARARĀJA'S CORRECTIONS FOR CIRCLING THE SQUARE

When a square of side s is transformed into a circle of equal area (approximately) by the usual and popular *Śulba* rule, as expressed by relation (7), the radius r of the equivalent circle will be

$$r = (2 + \sqrt{2})\, s/6 \qquad \ldots (31)$$

This gives a slightly higher value of r than the exact theoretical value mentioned in (3). However, this conclusion itself is based on using exact values of $\sqrt{2}$ and $\pi$. Obviously, we should try various approximations of these two irrational numbers as found in the ancient works.

For $\sqrt{2}$, the best approximation found in the texts of *Śulba Sūtras* is the following[14]:

$$\sqrt{2} = 1 + 1/3 + 1/(3.4) - 1/(3\,.\,4\,.\,34) \qquad \ldots (32)$$

$$= 577/408 \qquad \ldots (33)$$

$$= 1.4142157 \text{ nearly} \qquad \ldots (34)$$

This value itself implies the approximations 4/3 and 17/12 by taking the first two and three terms respectively. Other approximations found implied in the *Śulbas* include 7/5, 10/7, 36/25, etc.[15]. The comparative results obtained by using these values are given in Table 1 (in which the implied value of $\pi = s^2/r^2$).

Table 1

| S.No. | Value of $\sqrt{2}$ | Vaule of r/s | Implied value of $\pi$ |
|---|---|---|---|
| 1 | 4/3 | 5/9 = 0.556 | 81/25 = 3.240 |
| 2 | 7/5 | 17/30 = 0.567 | 900/289 = 3.114 |
| 3 | 10/7 | 4/7 = 0.571 | 49/16 = 3.063 |
| 4 | 17/12 | 41/72 = 0.569 | 5184/1681 = 3.084 |
| 5 | 36/25 | 43/75 = 0.573 | 5625/1849 = 3.042 |
| 6 | 577/408 | 1393/2448 = 0.5690 | $(2448/1393)^2$ = 3.088308 |
| 7 | exact $\sqrt{2}$ | $(2+\sqrt{2})/6$ = 0.5690 | $18(3-2\sqrt{2})$ = 3.088312 |
| 8 | Wanted value, $(6/\sqrt{\pi})-2$ | $1/\sqrt{\pi}$ = 0.5642 | $\pi$ = 3.1416 nearly |

That the prescribed *Śulba* rule (31) does not give accurate results was known to some commentators who tried to check it by applying it to particular cases. With the famous *Śulba* value (33), relation (31) gives

$$r = (1393/2448)\ s \quad \ldots (35)$$

To check the mathematical accuracy of this, Sundararája, in his commentary on *ASS*[16], takes *Saptavidha Agni* in the form of a *Rathacakraciti* or a circular altar of area 108000 square *aṅgulas*[17]. If this is converted into equivalent square-form, the side of the latter will be

$$s = \sqrt{108000} = \sqrt{[(328)^2 + 416]} \quad \ldots (36)$$

$$= 328 + 416/657 \quad \ldots (37)$$

by applying the well known ancient formula[18]

$$\sqrt{(a^2 + x)} = a + x/(2a + 1) \quad \ldots (38)$$

In fact, by using this rule for extracting approximate square-roots, the first two terms in Eq. (24) as well as in Eq. (32) can be successfully obtained.

Since one *aṅgula* was subdivided into 34 *tilas*, the side of the equivalent square will become

$$s = 328 \textit{ aṅgulas} + 416 \times 34/657 \textit{ tilas}$$

$$= 328 \textit{ aṅgulas} + (21 + 347/657) \textit{ tilas} \quad \ldots (39)$$

$$= 328 \textit{ aṅgulas} \text{ and } 21.5 \textit{ tilas} \text{ nearly} \quad \ldots (40)$$

as given by Sundararāja.

Now if we use rule (35) to convert the square of the above side, s, into a circle, its diameter will be given by

$$d = 2r = (1393/1224).\ (328 + 21.5/34)$$

$$= 374.007 \textit{ aṅgulas} \text{ nearly} \quad \ldots (41)$$

which Sundararāja gives as 374 *aṅgulas*. For finding the area of the circle with this d as diameter, he quotes the rule of Āryabhaṭa I, which amounts to using the value[19]

$$\pi = 62832/20000 \quad \ldots (42)$$

So, finally we will get the area of the circle (of above diameter 374) to be

$$= 62832 \times (374)^2/4 \times 20000$$

$$= 109858.6 \text{ square } \textit{angulas} \text{ nearly} \qquad \ldots (43)$$

which is given in the commentary as 109860 units.

Comparing this value with the original or expected value of 108000 units (with which he started), Sundararāja points out that there is an excess of 1860 units, thereby leading to contradiction or "*paraspara-virodhaḥ*", as he puts it. Actually, the excess arose due to the approximate nature of *Śulba* rule (31) or (35).

Hence, to improve the rule, Sundararāja prescribes the use of a correction. According to him, the rule will give accurate result by employing the correcting factor.

$$f_1 = (1 - 1/118) \qquad \ldots (44)$$

So that instead of Eqs. (31) and (35), we will have now

$$r = (2 + \sqrt{2})\, s\, f_1/6 \qquad \ldots (45)$$

and

$$r = (1393/2448)\, s\, f_1 \qquad \ldots (46)$$

The Sanskrit text for the correction is given by Sundararāja in the prose form in his commentary (p. 53) on *ASS* as well as expressed in the following verse found therein (p. 54):

चतुरश्रमण्डलकृतौ त्यक्तव्योऽष्टादशशतांशः ।
सूत्रोक्ताद्व्यासार्धाद् व्यासार्धं मण्डलस्यैतत् ॥1॥

*Caturaśra-maṇḍalakṛtau tyaktavyo'ṣṭādaśa-śatāṃśaḥ* ।
*Sūtroktād-vyāsārdhād vyāsārdhaṃ maṇḍalasyaitat* ॥1॥

'While converting a square into a circle, 118th part of the prescribed radius should be left out; the result is the radius of the desired (equivalent) circle'.

Hence, we get (45) with the correction factor (44).

When this correction is applied, the diameter of the circle equivalent to the square of side s given by (40), will be, by (46),

$$d = \frac{1393}{1224} \times (328 + \frac{21 \quad 5}{34}) \times \frac{117}{118}$$

$$= 374 \times 117/118 \text{ nearly}$$

$= 370$ *aṅgulas* and 28.24 *tilas* nearly

which is given by Sundararāja as

371 *aṅgulas* minus 6 *tilas*.

And the area of this circle will be

$$= (370 + 28/34)^2 \times 3.1416/4$$

$= 108000.4$ square *aṅgulas* nearly

which is practically same as the expected value, 108000 units (with which we started as the area of *Saptavidha Agni*).

The next verse given by Sundararāja (p. 54) contains an alternative rule, with correction, for circling the square. It can be expressed as[20]

$$d = (1 + 1/8)(1 + 1/330)\, s \qquad \ldots (47)$$

This rule implies a knowledge of (17) with the correction factor

$$f_2 = (1 + 1/330) \qquad \ldots (48)$$

To derive the numerical value (44), we follow Sundararāja in employing Āryabhaṭa's value of $\pi$ given by (42). Using the basic relation (2), we get from (46),

$$(62832/20000)\,.\,(1393/2448)^2\, f^2_1 = 1 \qquad \ldots (49)$$

or $\quad f_1 = (2448/1393)\,.\,\sqrt{(1250/3927)}$

$$= (2448/1393).\, 50/\sqrt{7854} \qquad \ldots (50)$$

Now by using the usual ancient formula (38), we have

$$\sqrt{7854} = \sqrt{(88^2 + 100)} = 88 + 110/177$$

$$= 88.62 \text{ nearly} \qquad \ldots (51)$$

Putting this in (50) and simplyfying, we easily get

$$f_1 = 122400/123447.66$$

$$= 1 - (1047.66/123447.66)$$

$$= 1 - 1/117.83 \text{ nearly} \qquad \ldots (52)$$

which gives the required value (44) by simply rounding off the figures.

If we apply the same method to find $f_2$ in improving (18) to

$$r = d/2 = (9/16)\ s\ f_2$$

we will have, like (50),

$$f_2 = (16/9)\ .\ 50/\sqrt{7854}$$

$$= 800/9 \times 88.62, \text{ by } (51)$$

$$= 800/797.58$$

$$= 1 + (2.42/797.58)$$

$$= 1 + 1/329.58 \text{ nearly}$$

which gives (48) by rounding off. Thus, we arrive at formula (47) by the same rationale.

## 4. Sundararāja's Corrections for Squaring the Circle

As mentioned above, the most popular *Śulba* rule for coverting a circle into an equivalent square is given by (23). Now Sundararāja, in his commentary on *ASS* which gives this rule, first checks its accuracy. He applies it to the case d = 374 *aṅgulas,* which we came across in Eq. (41). Now by using (23), the side of the equivalent square will be given by

$$s = 13 \times 374/15 = 324 + 2/15$$

$$= 324 \textit{ aṅgulas} \text{ and } (4 + 8/15) \textit{ tilas}$$

which has been taken by Sundararāja (p. 53) to be[21]

$$324 \textit{ aṅgulas} \text{ and } 4.5 \textit{ tilas} \qquad \ldots (53)$$

Comparing this with the expected value given by (40), he points out that the result obtained by the said rule (23) is in defect (*hīyanate*) by 4.5 *aṅgulas*[22].

Hence, here also Sundararāja prescribes a correcting factor which is

$$f_3 = (1 + 3/133) \qquad \ldots (54)$$

The Sanskrit text for the correction is found stated in prose as well as in a verse. The former reads (p. 54)

मण्डस्यापि चतुरश्रकरणे सूत्रोक्ते विष्कम्भे स्वस्मात् त्रिगुणात् त्रयस्त्रिंशच्छतांशं युञ्ज्यात् ।

*Maṇḍalasyāpi caturaśrakaraṇe sūtrokte viṣkambhe svasmāt triguṇāt trayastrimśacchatāṃśaṃ yuñjyāt*।

'In squaring a circle also, the side (here called *viṣkambha* or width) as obtained by the prescribed rule, should be increased by three times its own 133rd part.'

That is, we should take

$$s = (13d/15)\,(1 + 3/133) \qquad \ldots (55)$$

instead of (23).

Exactly the same correction is described in verse 3 (p. 54), which should be taken as [23]

मण्डलचतुरश्रकृतौ विष्कम्भे सूत्रचोदिते युञ्ज्यात् ।
त्रिगुणात्स्वकात्त्रयत्तिंरशच्छत भागं स विष्कम्भः ॥3॥

*Maṇḍacaturaśrakṛtau viṣkambhe sūtracodite yuñjyāt*।
*Triguṇātsvakāt trayastrīmśacchata bhāgam sa viṣkambhaḥ*॥3॥

'In squaring a circle the prescibed width should be increased by three times its own 133rd part, that becomes the (corrected) width (of the desired square)'

So that we have (55) here also.

In verse 4, Sundararāja gives the improved or corrected version of rule (27). The text should be taken to read as [24]

मण्डलविष्कम्भाद् द्वात्रिंशत्त्रिशतांशकं परित्यज्य ।
शिष्टान्नवमं जह्याच्चतुरश्रस्यैष विष्कम्भः ॥4॥

*Maṇḍala-viṣkambhād-dvātriṃśat-triśatáṃśakaṃ parityajya* ।
*Śiṣṭānnavamaṃ jahyāccaturaśrasyaiṣa viṣkambhaḥ* ॥4॥

'From the diameter of the given circle, leave out its 332nd part. The remainder should be decreased by its 9th part. That becomes the width (or side) of the desired (equivalent) square.'

That is,

$$s = (d - d/332)(1 - 1/9) \qquad \ldots (56)$$

$$= (8d/9).(1 - 1/332) \qquad \ldots (57)$$

which is equivalent to rule (27) with the correction factor

$$f_4 = (1 - 1/332) \qquad \ldots (58)$$

The most significant point to note is that the rationale of the correcting factors $f_3$ and $f_4$ is the same as that for $f_1$ and $f_2$. For instance, let the correct or improved version of (23) be

$$s = (13d/15)f \qquad \ldots (59)$$

Now by using the basic relation (2) with Āryabhaṭa's value of $\pi$, we get

$$2\ s/d = \sqrt{\pi} = \sqrt{(62832/20000)}$$

or, from (59)

$$26f/15 = \sqrt{7854}/50$$

This gives

$$f = 3\sqrt{7854}/260$$

$$= 3 \times 88.62/260, \text{ by } (51)$$

$$= 265.86/260$$

$$= 1 + 5.86/260$$

$$= 1 + 1/44.37, \text{ nearly}$$

$$= 1 + 3/133.11 \qquad \ldots (60)$$

which gives the prescibed correction (54) or $f_3$ by omitting the small fractional part in the denominator of the second term.

Similarly, let the correct version of (27) be

$$s = (8d/9)\, f \qquad \dots (61)$$

So that here we will have

$$16f/9 = \sqrt{7854}/50$$

or

$$f = 9\sqrt{7854}/800$$

$$= 9 \times 88.62/800, \text{ by } (51)$$

$$= 797.58/800$$

$$= 1 - 2.42/800$$

$$= 1 - 1/330.58, \text{ nearly} \qquad \dots (62)$$

Here, the denominator of the second term was taken conveniently as 332 (perhaps due to perference to even figure when the number is large), although by rounding off it comes out to be 331. Of course, this make a very minor difference in the correction.

Also, rule (47) simplifies to

$$d = (9/8).\,(331/330)\, s \qquad \dots (63)$$

So, the corresponding converse form of this rule will be

$$s = (8d/9)\,.\,(330/331)$$

$$= (8d/9)\,.\,(1 - 1/331) \qquad \dots (64)$$

which also shows that the denominator in the second term of $f_4$, given by (58), should be 331, instead of 332.

## 5. Summary and Concluding Remarks

*ASS* gives side by side the following usual and well-known Indian rules for circle-square conversions

$$d = (2 + \sqrt{2})\, s/3 \qquad \dots (65)$$

$$s = 13d/15 \qquad \dots (66)$$

Sundararāja, in his commentary on *ASS*, gave their improved or corrected forms as

$$d = (2 + \sqrt{2})\,(s/3)\,.\,(1 - 1/118) \qquad \ldots (67)$$

$$s = (13d/15).\,(1 + 3/133) \qquad \ldots (68)$$

He also gives the following pair of improved rules for the same purpose

$$d = (9s/8)\quad (1 + 1/330) \qquad \ldots (69)$$

$$s = (8d/9)\quad (1 - 1/332) \qquad \ldots (70)$$

We have shown above that the correction factors in all the improved forms (67) to (70) can be successfully obtained with the help of Āryabhaṭa I's value of $\pi$. Most probably this was the method actually followed.

Once the correction factors were chosen in suitable forms and applied, better results are expected. Each of the rules (67) to (70) will imply a value of $\pi$ which can be found by computing $4s^2/d^2$ or $s^2/r^2$ in each case. The results are presented in Table 2.

Table 2. Improved rules

| Sl No. | Rule | Implid value of $\pi$ |
|---|---|---|
| 1 | $d = \frac{(2 + \sqrt{2})}{3} . (1 - \frac{1}{118})\, s$ | $\frac{27848\,(3 - 2\sqrt{2})}{1521} = 3.141329$ |
| 2 | $d = \frac{1393}{1224} . (1 - \frac{1}{118})\, s$ | $\frac{(32096)^2}{(18109)^2} = 3.141325$ |
| 3 | $s = \frac{13}{15} . (1 + \frac{3}{133})\, d$ | $\frac{12503296}{3980025} = 3.14151$ |
| 4 | $d = \frac{9}{8} . (1 + \frac{1}{330})\, s$ | $\frac{3097600}{986049} = 3.14143$ |
| 5 | $s = \frac{8}{9} . (1 - \frac{1}{332})\, d$ | $\frac{1752976}{558009} = 3.14148$ |
| 6 | $s = \frac{8}{9} . (1 - \frac{1}{331})\, d$ | $\frac{3097600}{986049} = 3.14143$ |

| | | |
|---|---|---|
| 7 | $s = \frac{9785}{11136} \cdot (1 + \frac{1}{117})\, d$ | $\frac{(1154630)^2}{(651456)^2} = 3.14134$ |
| 8 | $s = \frac{9785}{11136} \cdot (1 + \frac{3}{266})\, d$ | $\frac{(2632165)^2}{(1481088)^2} = 3.15838$ |

It has already been mentioned that the special rule (29) is a peculiarity of *BSS* (the oldest of the *Śulbas)* and is not found in *ASS, KSS* and *MSS.* Perhaps its complicated form is responsible for its unpopularity. Moreover, the availability of rule (23) served as an alternative simpler method for squaring a circle.

It may be interesting to give rule (29) a treatment similar to what Sundararāja gave to (23). Now (29) simplifies to

$$s = (9785/11136)\, d \qquad \ldots (71)$$

We apply this to the case d = 374 *aṅgulas,* which we met in Eq. (41). The side of the equivalent square will be

$$s = 9785 \times 374/11136$$

$$= 328.63 \text{ nearly}$$

$$= 328 \textit{ aṅgulas} \text{ and } 21.4 \textit{ tilas} \qquad \ldots (72)$$

which is almost equal to the expected value given by (40). In fact, result (72) is far better than (53), which was obtained by applying (23) to the same case. Thus, Sundararāja would not have noticed any discrepancy which he noted while applying (23) in this way. Relation (71) implies

$$\pi = (9785/5568)^2$$

$$= 95746225/31002624$$

$$= 3.088326 \text{ nearly} \qquad \ldots (73)$$

This is better than the value (26) which is implied in (23). It must be noted that the accuracy of (71) is almost same as that of rules (31) and (35), as can be seen by comparing (73) with the corresponding values of $\pi$ in Table 1. In fact, rule (71) seems to represent the converse form of rule (31) or (35) whatever be its method of derivation. This is the reason why the value (72) is nearly equal to the original expected value of the side with which we started in (40), although rule (71) itself is only an approximate one.

Let us improve (71) by applying the correction factor f, as was done in the case of other formulas. That is, we take,

$$s = (9785/11136)\ d.f \qquad \ldots (74)$$

The value of f can be found by making (74) yield Āryabhaṭa's value of $\pi$, instead of (73). Thus, we must have

$$62832/20000 = (9785/5568)^2\ f^2$$

or

$$f = (5568/9785).\ \sqrt{7854/50}$$

$$= (2784/244628).\ \sqrt{7854}$$

$$= 2784 \times 88.62\ /\ 244628, \text{ by } (51)$$

$$= 1 + 2090.08\ /\ 244628$$

$$= 1 + 1/117.04, \text{ nearly}$$

$$= 1 + 1/117 \qquad \ldots (75)$$

by the usual rounding. This value of f again confirms that rule (71) is a sort of converse of (31) or (35). The corrected value (74) is also included in Table 2. It belongs to the same class of improved rules, although an ancient reference is yet to be found.

However, Datta[25] has mentioned the factor

$$f' = (1 + \frac{1}{2} . \frac{3}{133}) \qquad \ldots (76)$$

instead of (75) for correcting the *BSS* rule (29), or (30), or equivalently (71). He credits Dvārakānātha Yajva for giving correction (76) in his commentary on the *BSS*. But the referred passage in Dvārakānātha's printed commentary on *BSS* is almost a verbatim reproduction of the prose part of Sundararāja's corresponding commentary on *ASS* under the relevant rule[26]. Moreover, no Sanskrit text (sentence or passage) is found which contains a statement of (76). The Sanskrit sentence apparently relevant to (76) seems to be the same as we have already quoted, translated, and interpreted to contain rule (55) above[27]. Thus, according to our interpretation and finding, the relevant Sanskrit passage contains the factor $f_3$ given by (54) as a correction to rule (23), and not the above f'.

In fact, the so-called Dvārakānātha's commentary on the relevant *BSS Sūtras* contains only two corrections, which are same as those given by

Sundararāja in his commentary on *ASS*, which itself contains only the two rules. If Dvārakānātha was the author or originator of such corrections, one would expect, not two, but three corrections, since *BSS* contains three rules – (9), (23) and (29). Thus, the so-called Dvārakānātha's correction to the 'squaring the circle' refers to rule (23) and not to (29), and the correcting factor is (54) and not (76). There is no correction from him for (29), or (30), or (71), on similar lines as (75), which we have suggested by following the same ancient method[28]. If the said correction were meant for rule (29), there was no need of giving it in *ASS*, which does not contain this rule.

If one compares Sundararāja's commentary on *ASS* with that of Dvārakānātha on *BSS*, similarity of wording may be easily detected in respect of some passages. Careful study indicates that Sundararāja's commentary looks to be original and appropriate. For instance, regarding the passage

यावदिच्छं पार्श्वमान्या ------------------------------------------ प्रकारः ।

*Yāvadiccham pārśvamānyā* ........................................ *prakāraḥ* ।

which is given by Sundararāja (p. 49) as well as Dvārakānātha (p. 21), Datta remarks[29]:

"So one has doubtless copied from the other ....................................
Sundararāja explains the method .............................. (in) its proper place.
But Dvārakānātha ....................................... (calls it) 'another method'".

In fact, all evidences show that it is Dvārakānātha who quotes from Sundararāja, often with some changes and omissions, although Datta believed the contrary[30].

But who was this Sundararāja? Pingree[31] equates him with the famous Tamil astronomer Sundararāja (son of Anantanārāyaṇa), who was a contemporary of Nīlakaṇṭha Somayāji (*c.* AD 1500) and who commented on *Vākyakaraṇa*, a manual on astronomy[32]. However, our Sundararāja, the commentator of *ASS*, is said to be a son of Mādhavācārya[33], and so seems different from the above.

Regarding the time of our Sundararāja, some indications are there. From the dates mentioned in the post-colophons of certain manuscripts of his commentary on *ASS*, Datta points out that he lived before the end of the 16th century of our era[34]. Also, as we have seen above, Sundararāja has given correction for the rule

$$s = 8d/9$$

for the squaring of a circle. But this rule in its unimproved form continued to be found even in *Śulba Vārtika* (AD 1434) of Rāma Vājapeyin, as we have already mentioned in a conjectural way. So, if we suppose that the correction of the rule was done after AD 1434, Sundararāja may be placed between this date and about AD 1575. But we cannot be sure, and further investigation will be needed.

## References and Notes

1. Khadilkar, S.D.,(Ed.) *Kātyāyana Śūlba Sūtra,* with English translation, Poona, 1974, p.x.

2. Datta, B., *Science of the Sulba,* Calcutta, 1932, p. 6.

3. See *BSS,* 2.9 (p. 19), *ASS,* 3.2 (p. 40) *KSS,* 3.11 (p. 56) and *MSS,* 1.8 (p. 58) in S.N. Sen and A.K. Bag (editors), *The Śulbasūtras of Baudhāyana, Apastmba, Kātyāyana and Mānava* (with English translation), New Delhi, 1983. Unless otherwise stated, references to rules and pages are according to this convenient edition of the four *Śulbas.*

4. Takao Hayashi, A New Indian Rule for the Squaring of a circle etc. *Ganita Bhāratī* **12,** 75-82, 1990, especially p. 81.

5. Gupta, R.C., New Indian Values of $\pi$ from the *Mānava Śulba Sūtra, Centaurus (Denmark),* **31,** 114-125, 1988.

6. The use of $\pi = 25/8$ in Babylonian mathematics has been mentioned by Otto Neugebauer, *The Exact Sciences in Antiquity,* Harper, New York, 1962, P. 52, and by C.B. Boyer, "The History of Calculus" in the *N.C.T.M. 31st Yearbook,* Washington, 1969, p. 377.

7. Hayashi, *op. cit.* (ref. 4), pp. 77-80.

8. Sen and Bag (Eds.), *op. cit.* (see ref. 3), p. 164.

9. For details, see Gupta, *op. cit.* (ref. 5), pp. 116-117.

10. Chace, A.B., (Tr), *The Rhind Mathematical Papyrus,* reprinted by N.C.T.M., Reston, 1979, p. 49, Column 92.

11. Khadilkar, S.D. (Ed.), *op. cit.* (ref. 1), pp. 66 and 68.

12. *KSS* (Ed., G.S. Nene and A.S. Dogra) with the commentaries of Karka and Mahīdhara, Kashi Sanskrit Series No. 120, Benaras, 1936, p. 22.

13. David Pingree, Jyotiḥśāstra (Astral and Mathematical Literature) In Jan Gonda (Ed.), *A History of Indian Literature,* Vol. VI, Fasc. 4, Wiesbaden, 1981, pp. 6-7. Pingree mentions that Rāma wrote two more commentaries on *KSS* and cites a paper of S.L. Katre from *Proceedings of the All India Oriental Conference* **13,** 72-78, 1946.

14. See Gupta, R.C., Baudhāyana's Value of Root Two, *Mathematics Education,* **6,** Sec. B, 77-79, 1972, and On some Ancient and Medieval Methods of Approximating Quadratic Surds, *Ganita Bhāratī,* **7,** 8-15, 1985.

15. See Gupta, ref, 5, p. 121, and Datta, ref. 2, pp. 202-205.

16. See Srinivasachar, D. and Narasimhachar, Vidwan S. (Eds.), *ASS with the Commentaries of Kapardi, Karavinda and Sundararāja,* Mysore, 1933, pp. 53-54 which will be cited again and again.

17. *Saptavidha Agni* has area equal to 7.5 square *Puruṣa,* and 1 *purusa* = *120 aṅgulas.*

18. For details see Gupta's two papers mentioned above under ref. 14.

19. From the *Āryabhaṭīya,* II. 10, but without naming the source.

20. Here, we have adopted the correct reading *triṃśat-triśatāṃśa* (1/330) as given in the list of the variant readings (p. XVII) attached to the Mysore edition of *ASS* (ref. 16). The reason for accepting the above reading instead of *triṁśa-śatāṁśa* (1/130) will be clear from the rationale of the rule discussed in this paper itself.

21. The 4.5 *tilas* have been mentioned as *'ardhapañcamāśca tilāḥ'*.

22. The correct reading for this defect of 4.5 *aṅgulas* is to be found on p. xvii of the Mysore edition of *ASS* (see ref. 16), as *'sārdhāścatasro'*. Or we may correct the reading on p. 53 (line 18) to *'saptadaśatilāś-catasro'ṅgulayo'* (= 17 *tilas* and 4 *aṅgulas*).

23. We have slightly changed the second line of the verse in the light of the variant reading as given on p. xvii.

24. Here again, the correct reading, *dvātriṁśattriśatāṁśakam,* is adopted with the help of the variant reading given on p. xvii.

25. Datta, *op. cit.* (ref. 2), pp. 18 and 149.

26. Various editions of Dvārakānātha's commentary on *BSS* are available. The latest (and perhaps the best) is that by Pt. Vibhutibhusana Bhattacarya: *BSS with the Commentaries of Vyaṅkateśvara Dīkṣita and Dvārakānātha Yajva,* Sampurnanand Sanskrit University, Varanasi, 1979.

    Datta's referred passage is found on p. 26 of this edition and is a reprodution from the Mysore edition (ref. 16) of Sundararāja's commentary (prose part only, pp. 53-54) with some omissions and slight changes.

27. The Varanasi edition (ref. 26, p. 26) shows that Dvārakānātha changed the better phrase *Viṣkambhe svasmāt*, etc. (of Sundararāja) to simpler form, *viṣkambhas'ca syāt,* etc.

28. It was but natural that most of the modern scholars followed Datta in this point due to his great authority, e.g. see Sen and Bag (ref. 3), p. 179, and *Indian J. Hist. Science,* **15**, 151, 1980.

29. Datta, *op. cit.* (ref. 2), p. 89.

30. *Ibid,* p. 18.

31. Pingree, *op. cit.* (ref. 13), p. 6.

32. See Sarma, K.V. *A History of the Kerala School of Hindu Astronomy,* Hoshiarpur, 35, 56, 196, 1972.

33. See Raghavan, V., *New Catalogus Catalogorum,* Vol. II, Madras, p.131, 1966.

34. The *Bibliography of Sanskrit Works on Astronomy and Mathematics* (by S.N. Sen *et al*), New Delhi, 1966, pp. 210-211, also regards them different.

35. Datta, *op. cit.* (ref. 2), pp. 17-18.