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APPROXIMATE VALUES OF SURDS IN HINDU MATHEMATICS

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As has been shown in an earlier article, the Hindu interest in the athematics of surds is very old. The ancient Hindus were interested not only in the operations of the surds but also in finding their approximate values. The present article gives an account of the methods used for this purpose.

The method to find approximate values of surds is found as early as the time of the *Śulba*. Thus, Baudhāyana (800 BC) states:

"Increase the measure (of which the *dvi-karaṇī* is to be found) by its third part, and again by the fourth part (of this third part) less by the thirty-fourth part of itself (i.e., of this fourth part). (The value thus obtained is called) the *saviśeṣa*" (approximate)[1].

That is to say, if d be the *dvi-karaṇī* of a, that is, if d be the side of a square whose area is double that of the square on a then we shall have:

$$d = a + \frac{a}{3} + \frac{a}{3.4} - \frac{a}{3.4.34}, \text{approx.}$$

whence, we get

$$\sqrt{2} = 1 + \frac{1}{3} + \frac{1}{3.4} - \frac{1}{3.4.34}, \text{ approx.}$$

Expressing in decimal fractions, we obtain $\sqrt{2} = 1.4142156....$ According to modern calculation, $\sqrt{2} = 1.414213....$ Thus, it is clear that the ancient Hindus attained a very remarkable degree of accuracy in calculating an approximate value of $\sqrt{2}$. There has been much speculation among modern writers about the method by which the Hindus arrived at this result[2]. The most recent hypothesis is that of Bibhutibhusan Datta[3]. It is based on a simple and elegant geometrical procedure quite in keeping with the spirit of the early Hindu geometry and hence seems to be a very plausible one. According to Nīlakaṇṭha (*c.* 1500)[4], Baudhāyana supposed the side of a square to be 12 units in length, so that its diagonal would be $\sqrt{2 \, . \, 12^2} = \sqrt{288}$ units. Now $\sqrt{288} = \sqrt{17^2 - 1}$

$$= 17 - \frac{1}{34}, \text{ nearly}$$

Therefore $12\sqrt{2} = 17 - \frac{1}{34}$, nearly.

Hence, $$\sqrt{2} = \frac{17}{12} - \frac{1}{12.34},$$

or $$\sqrt{2} = 1 + \frac{1}{3} + \frac{1}{3.4} - \frac{1}{3.4.34}$$

Other notable approximate values occurring in the *Śulba* are[5]:

$$\sqrt{2} = \frac{7}{5},\ 1\frac{11}{25}, \qquad \sqrt{29} = 5\frac{7}{18},$$

$$\sqrt{5} = 2\frac{2}{7}, \qquad \sqrt{61} = 7\frac{5}{6}.$$

Probably it was also known that[6]

$$\sqrt{3} = 1 + \frac{2}{3} + \frac{1}{3.5} - \frac{1}{3.5.52}$$

In the early canonical works of the Jainas (500-300 BC)[7], we find applications of the formula

$$\sqrt{N} = \sqrt{a^2 + r} = a + \frac{r}{2a}$$

This formula has been applied consistently by the Jaina writers even up to the middle ages[8].

Bakhshālī Formula

In the Bakhshālī treatise on arithmetic (*c.* 200), we have the following rule for determining the approximate root (*śliṣṭa-mūla*, literally "nearest root") of a non-square number:

"In case of a non-square number, subtract the nearest square number; divide the remainder by twice (the root of that number). Divide half the square of that (that is, the fraction just obtained) by the sum of the root and fraction and subtract. (This will be the approximate value of the root) less the square (of the last term)[9]".

This is to say,

$$\sqrt{N} = \sqrt{a^2 + r} = a + \frac{r}{2a} - \frac{(\frac{r}{2a})^2}{2\,(a + \frac{r}{2a})}$$

approximately, the error being

$$\left[\frac{(\frac{r}{2a})^2}{2\,(a + \frac{r}{2a})}\right]^2$$

Example from the work:

$$\sqrt{41} = 6 + \frac{5}{12} - \frac{(\frac{5}{12})^2}{2\,(6 + \frac{5}{12})},$$

$$\sqrt{339009} = 579 + \frac{384}{579} - \frac{(384/579)^2}{2\,(579 + 384/579)}$$

In applying this approximate formula to concrete examples, the Bakhshali treatise exhibits an accurate method of calculating errors and an interesting process of reconciliation, the like of which are not met elsewhere[10].

LALLA'S FORMULA

To find the square-root of a sexagesimal fraction Lalla gives the following rule:

"Find the square-root (of the integral part in minutes) by the method indicated before. Multiply by sixty the remainder plus unity and then add the seconds. The result divided by twice the root plus 2 will be the fractional part (of the square root in terms of seconds)[11]"

That is if $\alpha = \beta^2 + \epsilon$, then we shall have

$$\sqrt{\alpha' r''} = \beta' + \left\{ \frac{60\,(\epsilon + 1) + r}{2\,(\beta + 1)} \right\}''$$

in sexagesimal fractions. The same formula appears in the *Rājamṛgāṅka of* Bhojarāja and the *Karaṇa-Kutūhala* of Bhāskara II[12]. It is obviously based on the approximate formula:

$$\sqrt{a^2 + r} = a + \frac{r + 1}{2\,(a + 1)}$$

BRAHMAGUPTA'S FORMULA

Brahmagupta (628) says:

"The integer (in degrees), multiplied by its sexagesimal fraction (in minutes) and divided by thirty is (approximately) the square due to the fraction which is to be added to the square of the integer[13]".

That is, we have

$$(\alpha^\circ \beta')^2 = (\alpha + \frac{\beta}{60})^2$$

$$= \alpha^2 + \frac{\alpha\beta}{30} + (\frac{\beta}{60})^2$$

$$= \alpha^2 + \frac{\alpha\beta}{30}, \text{ nearly,}$$

neglecting $(\beta/60)^2$ as being very small.

From the above rule, we easily obtain a formula for finding the approximate value of a non-square number. For if x be a small fraction compared with a, we have

$$(a + x)^2 = a^2 + 2\,ax$$

Putting $2\,ax = r$, we get

$$x = \frac{r}{2a}$$

Hence

$$\sqrt{a^2 + r} = a + \frac{r}{2a}$$

Brahmagupta expressly states a formula very much akin to that found in the Bakhshali treatise. To find the square-root of the sum or the difference of the squares of two numbers, the larger of which has a fractional part, he gives the following rule:

''Divide the square of the given smaller number plus or minus the portion in the square of the other due to its fractional part by twice (the integral part of) the other (at one place) and (at a second place) by the latter plus or minus the quotient obtained at the other place. The (last) divisor being added or subtracted by the last quotient and halved gives the square-root of the sum or the difference of the tv squares. Or it is the other number plus or minus that quotient''[1].

That is, if $a > b$ and ϵ, a small fraction, then

$$\sqrt{(a+\epsilon)^2 \pm b^2} = \frac{1}{2}\left\{2a \pm \frac{b^2 \pm (2a\epsilon + \epsilon^2)}{2a} \pm \frac{b^2 \pm (2a\epsilon + \epsilon^2)}{2a \pm \dfrac{b^2 \pm (2a\epsilon + \epsilon^2)}{2a}}\right\} \qquad \text{(i)}$$

or

$$\sqrt{(a+\epsilon)^2 \pm b^2} = a \pm \frac{b^2 \pm (2a\epsilon + \epsilon^2)}{2a \pm \dfrac{b^2 \pm (2a\epsilon + \epsilon^2)}{2a}} \qquad \text{(ii)}$$

The second formula gives an approximation by defect. The value

$$\sqrt{(a+\epsilon)^2 \pm b^2} = a \pm \frac{b^2 \pm (2a\epsilon + \epsilon^2)}{2a} \qquad \text{(iii)}$$

gives an approximation by excess. Taking the mean of (ii) and (iii), Brahmagupta finds the closer approximation given by (i).

On simplifying, we get from the formula (ii):

$$\sqrt{(a+\epsilon)^2 \pm b^2} = a \pm \frac{b^2 \pm (2a\epsilon + \epsilon^2)}{2a} \mp \frac{\left\{\dfrac{b^2 \pm (2a\epsilon + \epsilon^2)^2}{2a}\right\}^2}{2a + \dfrac{b^2 \pm (2a\epsilon + \epsilon^2)}{2a}}$$

Putting $\epsilon = 0$, $b^2 = r$, we have the formula

$$\sqrt{a^2 \pm r} = a \pm \frac{r}{2a} \mp \frac{(\frac{r}{2a})^2}{2a \pm \frac{r}{2a}}$$

Śrīdhara's Formula

Śrīdhara (*c.* 750) gives the following rule for finding the approximate value of the square-root of a non-square number:

"Multiply the non-square number by some large square number; then take the square-root (of the product), neglecting the excess, and divide it by the root of the multiplier"[14]

$$\sqrt{N} = \frac{\sqrt{N m^2}}{m} = \frac{R}{m}, \quad \text{nearly}$$

where m is an arbitrary large number and R is the nearest integral root of Nm^2. Śrīdhara gives two illustrative examples:

$$\sqrt{1000} = \frac{\sqrt{1000 \times 10000}}{100} = \frac{3162}{100} = 31\frac{31}{50}$$

$$\sqrt{6250} = \frac{\sqrt{6250 \times 10000}}{100} = \frac{7905}{100} = 79\frac{1}{20}$$

There are found various other formulae based upon Śrīdhara's formula. Thus, Āryabhaṭa II (*c.* 950) gives[15]:

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{ab \times 10000}}{b \times 100} = \frac{R}{b \times 100}$$

Śrīpati (1039) has[16]

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{ab \times m^2 \times 10000}}{b \times m \times 100} = \frac{R}{bm \times 100}$$

Bhāskara II (1150) states the formula[17]:

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{abm^2}}{bm} = \frac{R}{bm}$$

Example from Bhāskara II:

$$\sqrt{\frac{169}{8}} = \frac{\sqrt{169 \times 8 \times 10000}}{800} = \frac{3677}{800} = 4\,\frac{477}{800}.$$

Munīśvara (1658) gives

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{ab \times 10.000,000.000.000.000}}{b \times 100.000.000} = \frac{R}{b \times 100,000,000}$$

Illustrative example from him[19]:

$$\sqrt{208} = \frac{\sqrt{2080.000.000.000.000.000}}{100.000.000} = \frac{1442220510}{100.000.000}$$

$$= 14\,\frac{4222051}{10000000}$$

NĀRĀYAṆ'S METHOD

Nārāyaṇa (1356) says:

"Obtain the roots (of a square-nature) having unity as the additive and the number whose square-root is to be determined (as the multiplier). Then the greater root divided by the lesser root will be the approximate value of the square-root[20]"

That is to say, to find the approximate value of the surd $\sqrt{N}$ we shall have to solve the quadratic indeterminate equation

$$N\,x^2 + 1 = y^2$$

If $x = \alpha$, $y = \beta$ be a solution of this equation, then, says Nārāyaṇa

$$\sqrt{N} = \frac{\beta}{\alpha}, \text{ approximately.}$$

In illustration of his method, Nārāyaṇa finds approximations to $\sqrt{10}$ and $\sqrt{1/5}$[21]. Since the roots of $10 x^2 + 1 = y^2$ are (6, 19), (228, 721), (8658, 27379),......, we have

$$\sqrt{10} = \frac{19}{6}, \frac{721}{228}, \frac{27379}{8658}, \ldots.$$

Again the values of (x, y) satisfying the equation

$$\sqrt{\frac{1}{5}}\, x^2 + 1 = y^2$$

are (20, 9), (360, 161), (6460, 2889), ... Therefore

$$\frac{1}{5} = \frac{9}{20}, \frac{161}{360}, \frac{2889}{6460}, \ldots..$$

Jñānarāja's Method

Jñānarāja (1503) writes:

"Divide its square by the root of the nearest square number. The quotient together with that approximate root being halved will be a root more approximate than that. Values more and more accurate can certainly be found by proceeding in the same way repeatedly[22]".

In other words, if a^2 be the square number nearest to the non-square number N, so that $N = a^2 \pm r$, then the first approximate value (α_1) of $\sqrt{N}$ will be, says Jñānarāja,

$$\frac{1}{2}\left(a + \frac{N}{a}\right)$$

The next approximation will be

$$\frac{1}{2}\left\{\frac{1}{2}\left(a + \frac{N}{a}\right) + \frac{N}{\frac{1}{2}\left(a + \frac{N}{a}\right)}\right\}$$

and so on. The following illustrative examples are given:

$$\sqrt{8} = \frac{1}{2}\left(3 + \frac{8}{3}\right) = \frac{17}{6} = 2° \; 50' \text{ approximately}$$

$$\sqrt{8} = \frac{1}{2}\left(\frac{17}{6} + \frac{8 \times 6}{17}\right) = \frac{577}{204} = 2^\circ\ 49'\ 42'', \text{ approximately}$$

$$= \frac{1}{2}\left(2^\circ\ .\ 50' + 2^\circ\ .\ 49'\ 42''\right)$$

$$= 2^\circ 49' 51'', \text{ approximately.}$$

$$\sqrt{20} = \frac{1}{2}\left(4 + \frac{20}{4}\right) = \frac{9}{2} = 4^\circ 30', \text{ approximately}$$

$$= \frac{1}{2}\left(\frac{9}{2} + \frac{20 \times 2}{9}\right) = \frac{161}{36} = 4^\circ 28' 20'', \text{ approximately}$$

$$= \frac{1}{2}\left(4^\circ 30' + 4^\circ 28' 20''\right)$$

$$= 4^\circ 29' 10'', \text{ approximately.}$$

Formula of an Anonymous Writer

In his commentary on the *Līlāvatī* of Bhāskara II, Gaṇeśa (1545) has quoted a rule from a "previous writer" (*Ādya*) for finding the approximate value of the square-root of a non-square number.

It runs as:

"The residue of the root together with unity is multiplied by 60 and divided by twice the root plus 1. The sixtieth part of the root added with this fraction is (the required approximate value of) the root"

The process implied is clearly this:

$$\sqrt{N} = \frac{\sqrt{3600N}}{60}$$

Now on finding the square-root of 3600 N by the ordinary method for it, suppose the root comes out to be b and the residue in excess r. Then according to the rule

$$\sqrt{N} = \frac{1}{60}\left\{b + \frac{60\,(r + 1)}{2\,(b + 1)}\right\}$$

in sexagesimal fractions. It is obviously based on the approximate formula

$$\sqrt{a^2 + r} = a + \frac{r + 1}{2(a + 1)}$$

KAMALĀKARA

Kamalākara (1658) mentions all the formulae for finding the approximate value of a surd from that of Śrīdhara onwards[23]. But he has always employed the formula of Lalla. Its *rationale* has been given by him to be as follows[24].

Suppose

$$\sqrt{b^2 + r} = b + \epsilon$$

where ϵ is a small quantity. Then

$$b^2 + r = b^2 + 2b\epsilon + \epsilon^2$$

or $\epsilon (2b + 2\epsilon) = r + \epsilon^2$

Therefore, $\epsilon = \dfrac{r + \epsilon^2}{2b + 2\epsilon}$

$$= \frac{r + 1}{2b + 2} \text{ approximately}$$

Hence, we have the approximate formula.

$$\sqrt{b^2 + r} = b + \frac{r + 1}{2b + 2}$$

or in sexagesimal fractions:

$$\sqrt{b^2 + r} = b + \frac{60(r + 1)}{2(b + 1)}$$

Examples:

$\sqrt{5} = 2°14'10''$

$\sqrt{10} = 3°9'44''12'''$

$\sqrt{468°5'} = 21°28'7''$.

By the repeated application of the method, Kamalākara also finds the fourth root of numbers, e.g.,

$$\sqrt[4]{10} = 1°46'41''36'''$$

References and Notes

1. *Baudhayānā Śulba,* i. 61-2; see also *Āpastamba Śulba,* i. 6; *Katyāyana Śulba,* ii. 13.
2. Thibaut, *Śulvasūtras,* pp. 13 ff; C. Muller, "Die Mathematik der Śulvasūtra," *Abhand, a.d. Math. Sem.d. Hamburg Univ.,* Bd. vii, 1929, pp. 173-204.
3. Bibhutibhusan Datta, *Śulba,* pp. 192ff.
4. Vide his commentary on the *Āryabhaṭiya,*ii. 4. His commentary has been published in the Trivandrum Sanskrit series (Nos. 101, 110 and 185).
5. Datta, *Śulba,* p. 205.
6. For an elegant method of getting this approximate value see Datta, *Śulba,* pp. 194 ff.
7. For instance, see *Jambūdvīpaprajñapti, Sūtra,* 3, 10-16; *Jīvābhigamasūtra, Sūtra* 82, 124; *Sūtrakṛtāṅga-sūtra, Sūtra* 12, etc.
8. See the commentaries of Siddhasena Gani (*c.* 550), Malaya-giri (*c.* 1200) and others.
9. This rule is not preserved in its entirety at any place in the surviving portion of the Bakhshali manuscript; but it can be easily restored from the cross-references, especially on the folios 56, recto and 57 verso. See Bibhutibhusan Datta, *Bakh. Math.,* pp. 11 ff.
10. Datta, *Bakh. Math.,*pp. 14ff.
11. *ŚiDVṛ* iii. 52
12. *Rājamṛgāṅka* vi. 26 (c-d) – 28 (a-b); *Karaṇa-Kutūhala, spaṣṭādhikāra,* vs. 14.
13. *BrSpSi,* xii. 62.
14. *Tris (= Triśatikā),*R. 46.
15. *MSi = Mahā-siddhanta),* xv. 55.
16. *SiŚe (= Siddhānta-Śekhara),* xiii. 36
17. *L (= Līlāvatī),* p. 34.
18. *PāSā (= Pāṭīsāra),* R. 117.
19. *PāSā,* R. 120.
20. *NBi (= Nārāyaṇa's Bījagaṇita)* I, R. 88. Cf. Bibhutibhusan Datta, "Nārāyana's method for finding approximate value of a surd," *BCMS,* xxiii (1931), pp. 187-194.
21. *NBi,* I, Ex. 45
22. *Āsannamūlena hṛtāt svavargāl labdhena mūlaṁ sahitaṁ dvibhaktam* I
Bhavedāsannapadaṁ tato'pi muhurmuhuḥ syāt sphuṭamūlamevam II
Sundara-Siddhānta, bījādhyāya, 12 (c-d)-13(a-b).
23. *SiTVi,* iii. 10-19.
24. *SiTVi,* xiv. 324 (com).