Indian Journal of History of Science, **30**(1), 1995

POLYGONAL APPROXIMATION TO CIRCLE AND MĀDHAVĀCĀRYA

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(Received on 28th February 1994; after revision 7 November 1994)

The aim of this article is to study the Geometrical method of 'polygonal Approximation to Circle'[1] due to Mādhavācārya (1340 AD to 1425 AD)[2] *vis-a-vis* the commentary thereon by Śaṅkara Pāraśava (1500 AD to 1560 AD) in *Kriyākramakarī*,[3] a Kerala-based commentary on *Līlāvatī*.

Key Words: Mādhavācārya, polygonal approximation, Circumference, diameter, Geometrical Methodology, Āryabhaṭa, *Tiloyasāra*, Bhāskara, Nilkaṇṭha Somayajī, *hāra, hāraka, Koṭikaraṇāntara, bhujaikadesah, khalvihastasrasya, bhujaiyaḥ, Karṇarūpinyā, bhujakoṭyoranyatara, atastattulyaiva, taditarapi, tulyakaram, karṇasyobhayapār śvagatametad bhūmikam, abādhāntaram, abādhā, sādhāraṇī alpābādhā, koṇesvāsphālitasūtramadhya, kṣetrakendra, abādhāntaram, yāvadabhiṣṭam, sūkṣmatāmapādayitum, sakyam, Vṛttaprayam, Sulbā Sūtra, yuktibhāṣā, Āryabhaṭiyabhāṣya.*

Mādhavācarya (1340 AD 1425 AD),[2] a mathematician and astronomer, residing at the village of Saṅgamagrāma near Cochin in South India adopted Geometrical method to obtain a non-terminating series for representing the circumference (C) of a circle in terms of its diameter (D) namely:

$$C = \left[\frac{4D}{1} + \frac{4D}{5} + \frac{4D}{7} + \ldots\right] - \left[\frac{4D}{3} + \frac{4D}{7} + \frac{4D}{11} + \ldots\right] \text{verses 1 to 4.}^3$$

His invention of incommensurability of Paridhisaṃkhyā placed him in a very remarkable position in the history of the development of the concept of π. The present article will throw light on Mādhava's Geometrical Methodology leading to 'Polygonal Approximation to Circles' which has profoundly influenced the development of Mathematics.

'Polygonal Approximation to Circle'[4] involves construction of regular polygons of sides n ($n \geq 4$) each either circumscribing a circle or inscribed in it. The process starts with the construction of a regular polygon with minimum number of sides circumscribing a given circle or inscribed in it and the number of sides of the polygon is successively doubled until the sides of the polygon ultimately coincide with the circumference of the circle. Mādhava's method in the form of four verses [S_1, S_2, S_3, S_4, Appendix] has been included in the chapter on 'Relation between

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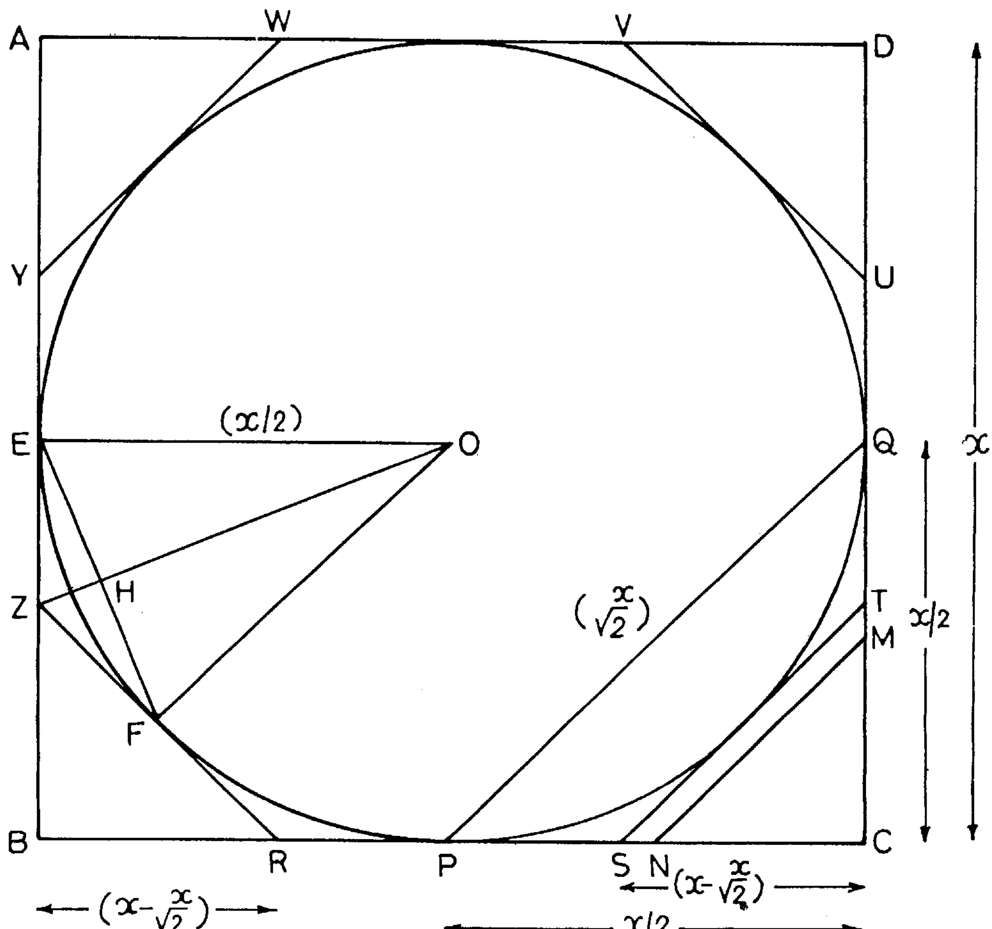
A
W
V
D
Y
U
E
$(x/2)$
O
Q
x
Z
H
$(\frac{x}{\sqrt{2}})$
T
$x/2$
M
F
B
R
P
S
N
C
$(x-\frac{x}{\sqrt{2}})$
$(x-\frac{x}{\sqrt{2}})$
$x/2$

Fig. 1

diameter and circumference' of a circle (*Vyāsaparidhyoḥ sambandhaḥ*)[5] in *Kriyākramakarī*.

The verses state that

> The square-root of one-eighth part of the square of side of a quadrilateral (actually, a square) is hāra. Hāra less one-fourth of the side is multiplied by the side and (the product is) divided by hāra. Taking segments equal the result thus obtained along a side from its corners (i.e taking segments equal in length to this result on every side from the corners on it) the octagon is formed.
> Adding the square of half of a side of the octagon to the square of the radius, the square-root (of the sum obtained here) is (called) the diagonal. By that diagonal is divided the square of the radius less half the square of a side (of the octagon). The result is subtracted from the diagonal and (the difference) is halved to get what is called *hāra;* the diagonal less held the diameter is multiplied by half of a side (of the octagon); the product (on being) divided by *hāra* gives
> Which is taken (for length of segments) along sides (measured) from corners (*i.e.*, on every side from either of it) and cut off to get a (regular) polygon of sixteen sides. In this way of cutting portions of each side (in between the corners on it) ultimately a circle can be obtained. (*cf.* verses 1-4).

Clarification by Śaṅkara Pāraśava

To clarify the above verses Śaṅkara started with a square whose one side (of length x, say) equals the diameter of a given circle (this indicates that the square circumscribes the circle). He, then proceeds to define

(i) the square-root of 1/8th part of the square of a side i.e., $\sqrt{\frac{x^2}{8}}$ as *hāraka*;

(ii) the *hāraka* less 1/4th of a side i.e., $\sqrt{\frac{x^2}{8}} - \frac{x}{4}$ as *koṭikarṇāntara*

(iii) the *koṭikarṇāntara* multiplied by a side and (the product) divided by *hāra* (or *hāraka*) i.e.

$$C = \frac{x\left(\sqrt{\frac{x^2}{8}} - \frac{x}{4}\right)}{\sqrt{\frac{x^2}{8}}} \quad \text{as } bhujaikadeśaḥ$$

Śaṅkara points out that this *bhujaikadeśah* is one of the two (equal) sides of the right-angled triangle whose hypotenuse is a side of the octagon (*sā khalvihāṣṭaśrasya bhujāyaḥ karṇarūpinyā bhujakoṭyoranyatarā atastattulyaiva taditarāpi*).

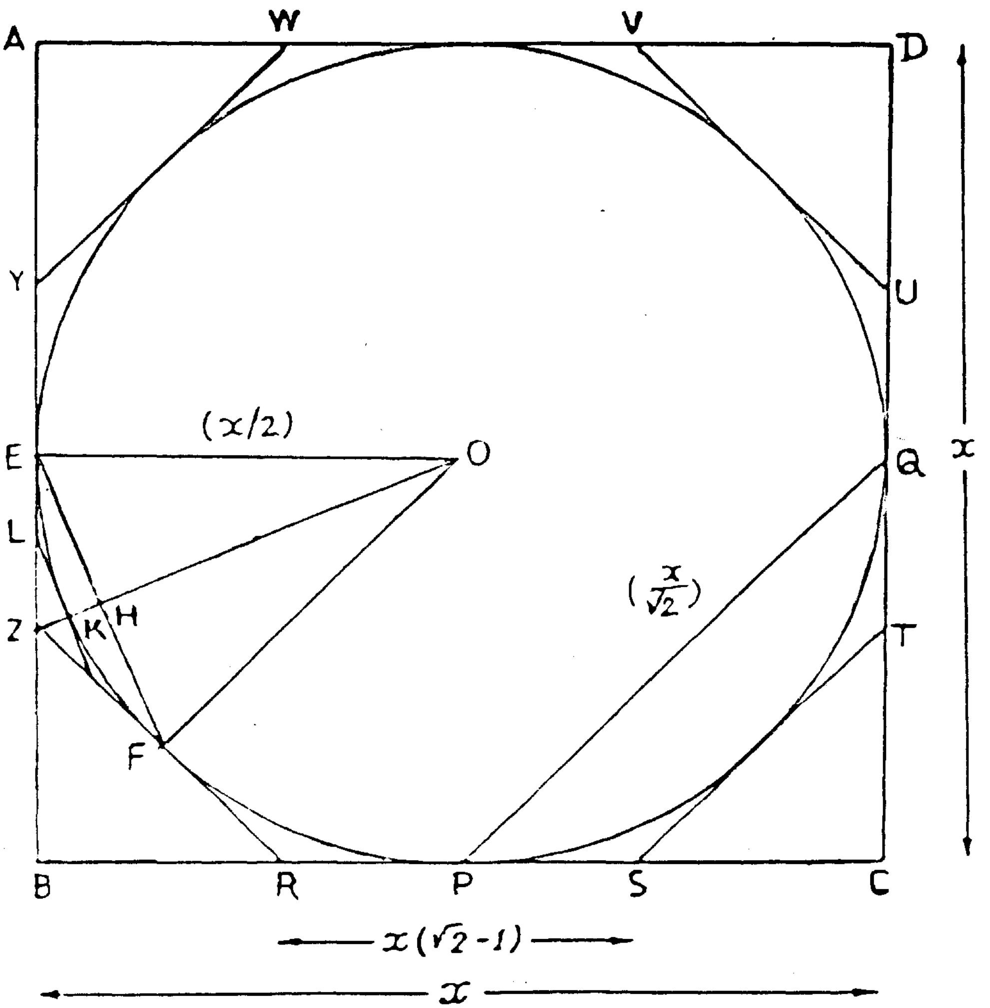

Fig. 2

Taking a piece of thread equal in length to this *bhujaikadeśah*, Śaṅkara said, 'Tie it to a corner of the square and stretch it along one side (of the square).' Four such pieces of threads are stretched so (along the four sides of the square). Bringing back the threads in their respective original positions the process is repeated (in opposite orientations). The length of a side (of the square) less twice the portion equal to this result (*i.e.*, the length of a side not covered by threads streched along it from its two ends) is a side of the regular octagon (circum scribing the circle). Thus, four sides of the octagon and from *bhujaikadeśah* (as twice the square of *bhujaikadeśah* is the square of a side of the octagon) four other sides are obtained.
i.e., if ABCD be the square with a side x unit long (and circumscribing a given circle of diameter x unit) then

$$MN = \textit{hāraka} = \sqrt{\frac{x^2}{8}},\ MC = CN = \frac{x}{4},\ \textit{koṭikarṇāntara} = MN = MC,$$

$$\textit{bhujaikadeśah} = \frac{x\left(\sqrt{\dfrac{x^2}{8}} - \dfrac{x}{4}\right)}{\sqrt{\dfrac{x^2}{8}}} = x - \frac{x}{\sqrt{2}} = BR = CS;$$

So that a side RS of the regular octogon (circumscribing the circle) is given by RS = BC - 2BR. Thus the four sides RS, TU, VW and YZ are obtained. Each of the remaining four sides can be obtained from the relation : square of such a side = 2x square of *bhujaikadeśah*.

Following Mādhava, Śaṅkara designates the segment joining the centre of the circle and a vertex of the octagon as a diagonal and (considering that the radius is perpendicular to a side of the octagon at the point where it touches the circle) gives that for a diagonal

OZ, $OZ = \sqrt{[EZ^2 + OE^2]}$, where EZ = 1/2 YZ and YZ = a side of the octagon.

Now, for each of two congruent (*tulyākāraṃ*) triangles OEZ and OZF standing on the common base OZ and lying on opposite sides of it (*karṇasyobhayapārśvagatametadbhūmikaṃ*) the radius bisecting a side of the octagon is one side and half of a side (of the octagon) extending from the foot of the perpendicular to the end of diagonal as the other side. Thus, there are two (congruent right-angled) triangles on the diagonal as the common base and lying on opposite sides of it (the common base). Then by the Rule 165 of the *Līlāvatī* on 'the sum of two sides of a triangle' [i.e., from EZO, $\dfrac{EO^2 - EZ^2}{OZ}$ = HO-ZH], *abādhāntaram* (HO-ZH) is obtained.

The diagonal (OZ) less then *abādhāntaram* is halved to obtain the common smaller *abādhā* (*sādhāraṇī alpābādhā*) ZH (the segments of the base of a triangle divided by the perpendicular from the vertex are called *abādhās*) and ZH = ½ [OZ – $\frac{EO^2 - EZ^2}{OZ}$].

There are two other triangles EZH and ZHF lying on opposite sides of the diagonal. EH and HF, both perpendicular to the diagonal, are respectively their bases and the *alpābādhā* ZH found here, is their height. Then, by the rule of three, to find which length (say, y) bears the same ratio to the diagonal-less-radius ie to OZ-OK (=ZK) as half of the side of the octagon (i.e., XY) bears to the perpendicular (ZH) *i.e.*, to find y where y:ZK = ZE:ZH where y=ZL.

Then, 'tie a thread of length ZL to each corner of the octagon and stretch it along the sides (in two orientations as before)'. The portions of sides not covered by the threads in two orientations are the eight sides of the (regular) polygon of 16 sides and from *bhujaikadeśaḥ* (= ZL) the remaining eight (oblique) sides are obtained. In this way, cutting segments from the sides of the 16-sided (regular) polygon, as the distance between the mid-point of a side (*koṇeṣvāsphālitasūtramadhya*) and the circumcentre (*kṣetrakendra*) is equal to the radius. Thus a (regular) polygon of thirty-two sides can be obtained. Then doubling the number of sides almost a circle can be had. Thus one can attain the desired accuracy.

i.e., from figure 2, a side of the regular octagon

$$= x - 2\left(x - \frac{x}{\sqrt{2}}\right) = x\,(\sqrt{2} - 1)$$

We have, EZ = ½ x (√2 – 1)

$$\text{So, } OZ^2 = EZ^2 + EO^2 = \frac{x^2}{4}(\sqrt{2}-1)^2 + \frac{x^2}{4} = \frac{x^2}{4}(4-2\sqrt{2})$$

$$\therefore OZ = \frac{x}{2}\sqrt{4 - 2\sqrt{2}}$$

abādhāntaram HO - ZH = $\dfrac{EO^2 - EZ^2}{OZ}$

alpābādhā ZH = $\frac{1}{2}\left[OZ - \dfrac{EO^2 - EZ^2}{OZ}\right] = \dfrac{EZ^2}{OZ} = \dfrac{x^2/4\ (\sqrt{2}-1)^2}{x/2\ \sqrt{4-2\sqrt{2}}} = \dfrac{x}{2}\ \dfrac{(\sqrt{2}-1)^2}{\sqrt{4-2\sqrt{2}}}$

And ZK = OZ – OK = $\dfrac{x}{2}\sqrt{4-2\sqrt{2}} - \dfrac{x}{2} = \dfrac{x}{2}\left[\sqrt{4-2\sqrt{2}}-1\right]$

∴ by rule of three, ZL = $\dfrac{ZK}{ZH}$. EZ gives

$$ZL = \frac{x}{2}\left[\sqrt{4-2\sqrt{2}}-1\right] . \frac{x}{2}(\sqrt{2}-1) . \frac{2}{x}\ \frac{\sqrt{4-2\sqrt{2}}-1}{(\sqrt{2}-1)^2}$$

$$= \frac{x}{2} \frac{(\sqrt{4-2\sqrt{2}}-1)\,(\sqrt{4-2\sqrt{2}})}{\sqrt{2}-1}$$ = *bhujaikadeśaḥ* of sixteen-sided (regular) polygon,

and a side of this ploygon is given by

$$x(\sqrt{2}-1) - 2 \,.\, \frac{x}{2} \frac{(\sqrt{4-2\sqrt{2}}-1)\,(\sqrt{4-2\sqrt{2}})}{(\sqrt{2}-1)}$$

$$= [\frac{(\sqrt{2}-1)^2 - (4-2\sqrt{2}) + \sqrt{4-2\sqrt{2}}}{\sqrt{2}-1}]$$

$$= [\frac{\sqrt{4-2\sqrt{2}}-1}{\sqrt{2}-1}] \quad \text{unit}$$

Verification of the Result

As every side of a regular octagon subtends $\frac{360°}{16}$ i.e. 45° at the centre of the circle circumscribed by the octagon,

$$\therefore \text{ a side of the octagon } = 2 \,.\, \frac{x}{2} \tan .\, 22\tfrac{1}{2} \text{ unit}$$

$$= x\,(\sqrt{2}-1) \text{ unit, where } x = \text{diameter of the circle.}$$

Also, a side of a regular polygon of 16 sides circumscribing a circle of diameter x substends an angle measuring $\frac{360}{16}$ degrees of 22½° at the centre of the circle

$$\therefore \text{ if } \tan\ (\frac{45°}{4}) = t, \text{ then } \sqrt{2}-1 = \tan 22\tfrac{1}{2}° = \frac{2t}{1-t^2} \Rightarrow (\sqrt{2}-1)t^2 + 2t - (\sqrt{2}-1) = 0$$

The positive root of this quadratic equation is given by

$$\frac{-2 + 2\sqrt{4-2\sqrt{2}}}{2(\sqrt{2}-1)} = \frac{\sqrt{4-2\sqrt{2}}-1}{\sqrt{2}-1}$$

So, a side of the regular sixteen-sided polygon (circumscribing a circle of diameter x unit) = $x \,.\, \frac{\sqrt{4-2\sqrt{2}}-1}{\sqrt{2}-1}$ unit, which agress with the corresponding result obtained geometrically.

Discussion

The above geometrical details of Mādhava's method of Polygonal Approximation to Circle (based on Śaṅkara Pārāśava's commentary) give all the step-by-step methods of construction of regular polygons with even number of sides together with necessary justification revealing the art of geometrical reasoning with pedagogical approach of the then India.

Though it is not explicitly mentioned in the verses, it is clear that the square and all the polygons in the subsequent cases each circumscribes a given circle. The methods of construction do not follow the Euclidean pattern[6] of drawing tangents to the circle at the vertices of the corresponding inscribed polygons.

Mādhava started with the construction of polygons each of which, infact circumscribed a given circle and Śaṅkara Pārāśava pointed out that due to the repeated doubling of the number of sides, the perimeter of the polygon, as it were, would move closer to the circumference and an approximation as close as required to the shape and size of the circumference can be obtained (*yāvadabhīṣṭaṃ sūkṣmatāmāpādayituṃ śakyam*) instead of accepting exact coincidence of the perimeter with the circumference of the circle at the ultimate stage (which justifies the statement : *vṛttaprāyam sādhayet*). Here in lies the speciality of the idea over the Western idea of exhaustion,[7] which basically states that if a rectilinear figure (a polygon) be inscribed within (instead of circumscribing, as in Mādhava's method) a curvilinear figure (a circle) and if the number of sides of the polygon be increased (by doubling the number of sides of the polygon) the area between them would finally be exhausted. This idea of exhaustion originated in Antiphone (430 B.C.),[8] flourished in Eudoxus (355 B.C.)[9] and practised by Archimedes (287 B.C to 212 B.C)[10] had been almost a customary process in the matter even upto the times of James Gregory (1630 A.D. to 1675 A.D.).[11] Mādhava's process was rather related to rectification of the circle and not to its quadrature.

Mādhava's method of 'Polygonal Approximation to Circle' finally aiming at obtaining a relation between the circumference (C) of a circle to its diameter (D) was unprecedented in India also. Approximate values of C in terms of D[12] (excepting those due to Mādhava) obtained from geometric consideration were mostly based on the construction of a single circle of known diameter. This will be more evident when the contributions of the *Śulba Sūtras*[13], of Āryabhaṭa I,[14] of Mādhavacandra[15] (1000 AD) (commentator of Nemaicandra's *Tiloyasāra*) are considered, Bhāskara II in his *Līlāvatī* prescribed rules[16] for numerical computation of a side of regular polygons (upto nine-sided) inscribed in a circle; but he did not turn towards 'Polygonal Approximation to Circle' through stages of increasing the number of sides of inscribed polygons. T.A. Sarasvatī Āmma[17] has pointed out that the *Yuktibhāṣā* of Jyeṣṭhadeva (1475 AD – 1575 AD)[18] employed the construction of polygons circumscribing a given circle starting from a square and proceeding upto a polygon with a very large number of sides. But this does not challenge Mādhava's seniority in the field. Mādhava's motivation aiming at the approximation of C in terms of D to any degree of accuracy brought an air of novelty in the world of Mathematics, he was reared in, and paved the way of advanced thinking in the line for later mathematiceans like Nīlkaṇṭha Somayāji (1443 AD to 1543 AD) who referred to Mādhava in the *Āryabhaṭiyabhāṣya*[19]

Findings

Mādhava's Method: For a circle of a given diameter (x) let a square with a side (x) be drawn. Let C denote the *bhujaikadeśaḥ* given by

$$C = \frac{x\left(\sqrt{\frac{x^2}{8}} - \frac{x}{4}\right)}{\sqrt{\frac{x^2}{8}}}$$

Measuring segments, each of length C along each side from the two ends and cuting them off, a side of a regular octagon can be obtained. Similarly by calculating *bhujaikadeśaḥ* (d) of this octagon a regular 16-sided polygon can be obtained.

Procedural Variance From Euclid

Though it is not explicitly mentioned in the verses it is clear that the square drawn at the outset and the regular polygons in all subsequent cases each circumscribes the given circle. The mode of construction do not follow Euclidean[20] way of drawing tangents to the circle at the vertices of the corresponding inscribed polygons.

Departure From The Western Idea

Unlike the Western idea of 'exhaustion[21]' wherein inscribing of polygons in a given circle was almost a customary process, Mādhava started with the construction of polygons which, infact, circumscribed the given circle and as Śaṅkara Pāraśava points out, that due to repeated doubling of the number of sides the perimeter of the polygon, as it were, would move closer to the circumference and an approximation as close as required to the shape and size of the circumference can be obtained. (*Yāvadaviṣṭam Sūkṣmatāmāpādayitu, śakyam*). So Mādhava's process was rather related to the rectification of a circle and not to its quadrature.

Appendix

S-1 चतुर्भुजे दोःकृतिनाग (८) भागमूलं हरो, हारभुजाङ्घ्रिभेदात्।
भुजाहताद्धारहृतं तु कोणान्नीत्वा विलिख्याष्टभुजाः प्रसाध्याः।। 1।।

S-2 अष्टाश्रदोरर्धकृतिर्निधेया व्यासार्धवर्गे पदमत्र कर्णः।
तेनाऽऽहरेद् दोर्दलवर्गहीनं व्यासार्धवर्गं यदतः फलं स्यात्।। 2 ।।
S-3 तदूनकर्णो दलितो हराख्यो गुणस्तु विष्कम्भदलोनकर्णः।
भुजार्धमेतेन हतं गुणेन हरेण भङ्क्त्वा यदिहापि लब्धम्।। 3।।
S-4 तत्कोणतः पार्श्वयुगेषु नीत्वा छिन्नेऽन्तरे स्यादिह षोडशाश्रम्।
अनेन मार्गेण भवेदतश्च रदाश्रकं बृत्तमतश्च साध्यम्।। 4।।

अस्य श्लोकचतुष्टयस्यार्थः तत्रादौ तावदिष्टव्याससमचतुर्भुजं क्षेत्रं परिकल्प्य तद्भुजावर्गादष्टभिर्विभज्य यल्लब्धं तस्य मूलं हारको नाम। ततस्तस्मादेव हारकात् पृथक्स्थिताद् भुजाचतुरंश विशोध्य यच्छिष्टं तेन भुजा निहत्य पृथक्स्थितेन हारेण विभजेत्। तत्र लब्धो भुजैकदेशो नाम। सा खल्विहाष्टाश्रस्य भुजायाः कर्णरुपिण्या भुजाकोट्योन्यतरा। अतस्तत्तुल्यैव तदितरापि, यतस्तयोर्वर्गयोगमूलमष्टाश्रभुजा। तदुक्तम् – 'कोणान्नीत्वा विलिख्याष्टभुजाः प्रसाध्याः' इति। तत्र 'दोःकृतिनागभाग'–स्तावद् "दोरर्धवर्गस्यार्ध, दोरर्धवर्गस्य दोर्वर्गचतुरंशत्वात्, तदर्धस्य तत्कृतिनागभागत्वात्। अन्यथां दोःकृतिनागभागस्तच्चतुरंरश वर्गो द्विगुणितश्चतुरंशवर्गस्य तद्वर्गपोडशांशत्वाद्, द्विगुणितस्य तस्य तद्वर्गाष्टांशत्वात्। ततस्तस्य यन्मूलं स दोश्चतुरंशतुल्ययोर्भुजाकोट्योः कर्णरुप एव, तयोवर्गयोगमूलरुपत्वान्। दो र्वर्गषोडशांशयोर्द्वयोर्योग एव हि तदष्टांशः। दोर्वर्गषोडशांशश्च तच्चतुरंशवर्गः। अतो दोश्चतुरशतुल्ययो र्भुजाकोट्योः कर्णरुप एवासौ। स एव च हारकः।

ततस्तस्य हारस्य कर्णरूपस्य भुजाचतूरंशस्य तत्कोटिरुपस्य च यो भेदः कोटिकर्णान्तरं नाम तेन चतुरश्रभुजां निहत्य हारेण विभज्य लब्धं यत् फलं तत् कोणतः पार्श्वयोर्नीत्वा तदग्रे सूत्रमास्फालयेत्। ततस्तथास्फालितानि चत्वारि सूत्राणि। चतस्त्रो भुजाः, पूर्वप्रदर्शितचतुर्भुजैकदेशाश्चतस्त्री भुजाः इत्यष्टो भुजाः साधिताः स्यु। ते पुनस्तथानीता अत्र लब्धेन फलेन द्विगुणितेन हीनं यच्चतुर्भुजमान तत्तुल्यप्रमाणाः स्युः।

अथ तदर्धस्य व्यासार्वस्य च वर्गयोगं मूलीकुर्यात्। लब्धं क्षेत्रकेन्द्रादष्टाश्रकोण प्रापी कर्णो भवति। अथास्य कर्णस्योभयपार्श्वगतमेतद्भूमिकं मिथस्तुल्याकारं क्षेत्रद्वयं कल्पनीयम्। तद्यथा–तत्र केन्द्रादष्टाश्रदोर्मध्याप्रापि यद् व्यासार्ध सैका भुजा, यच्च तत्सम्पातात् कर्णाग्रावधि दोरर्धं सात्या। एवं कर्णोभयवपार्श्वगतयोस्त्र्यश्रयोर्भुजौ। कर्णस्तु साधारणी भूरिति। तत्र 'त्रिभुजे भुजयोर्योगः' (लीला. 165) इति न्यायेन भुजावर्गाबाधा' – योगात्मकेनात्रानीतेन कर्णेन यल्लभ्यते तद् आबाधान्तरम्।

ततस्तदूनशिष्टस्य कर्णस्यार्धं त्र्यश्रयोः साधारणी अल्पाबाधा। अथान्यदपि त्र्यश्रं क्षेत्रं विद्यते। तद्यथा–कर्णोभयतःस्थितयोस्त्र्यश्रयोयौ लम्बौ तावाबाधे। ततस्तयोर्योगो भूमिः। अत्रानीताऽल्पाबाधा लम्बः। पूर्वं प्रकाल्पितयोस्त्र्यश्रयोरल्पभुजावेव भुजौ। ततस्तै स्त्रैराशिकम्–यद्यावाधात्मनानीतस्य लम्वस्याष्टाश्रदोरर्धात्मिके भुजे ततो व्यासार्धोनकर्णशेषस्य लम्वाग्रस्य कियत्यौ भुजे इति।

तत्र लब्धमष्टाश्रकोणतः पार्श्वयोर्नीत्वा तदग्रे सूत्रमास्फालयेत्। ततस्तावष्टौ, अष्टाश्रभुजैकदेशाश्चाष्टौ इति षोडशभुजास्साधिताः स्यूः। अथानेन न्यायेन पोडशाश्रकोणानपि

तथा छिन्द्याद् यथा कोणेष्वास्फालितसूत्रमध्यस्य क्षेत्रकेन्द्रस्य चान्तरालं व्यासार्धतुल्यं भवति। एवं द्वात्रिंशदश्रं साधयेत्। अथ ततस्ततो द्विगुणिताश्रेण वृत्तप्रायं साधयेत्। एवं यावदभीष्टं सूक्ष्मतामापादयितुं शक्यम् इति।

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