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A STEP TOWARDS INCOMMENSURABILITY OF Π AND BHĀSKARA (I) AN EPISODE OF THE SIXTH CENTURY AD

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The aim of this article is to investigate how Bhāskara I (round about 574 AD[1]) attempted to establish his views (geometrically) in defiance of the then practice of taking $\sqrt{10}$ D as the measure of the circumference (C) of a circle with diameter D and also the judgement he put forward on the basis of his ocular evidence (*pratyakṣeṇa*) that $\frac{c}{d}$ (= π, in modern notation) is incommensurable as well as non-constructible. This judgement, which we call Bhāskara (I)'s conjecture, essentially bears with it the modern concept that π is transcendental.

Key words : Āsannaparidhiḥ, Daśakaraṇiparidhiprakriyā, Evāgamaḥ, Karaṇisamāsa, Pratyakṣeṇa, Rāśyorsaṃkṣepatā, Transcendental.

Introduction

Since the times of *Sūryaprajñapti*[2] (a Jaina work containing mathematical and astronomical ideas flourished in India around 500 BC[3]) to the times of Śrīpati[4] (1039 AD), it was almost a customary affair in India as well as in China (Ch'ang Höng[5] : 75 AD to 139 AD) to take($\sqrt{10}$ D) as the measure of the circumference (C) of a circle of diameter (D).. Āryabhaṭa (I) (born in 476 AD) was the first to differ at this point. The verse [S - 1]** he put forward in the chapter 'Relation between diameter and circumference in a circle' (*vṛtte vyāsaparidhisambandhaḥ*) in the Mathematics Section (*Gaṇita Pādaḥ*) of his *Āryabhaṭīya*, prescribed that for a circle with D = 2000 unit, *C* is approximately 62832 unit. Bhāskara (I) in his commentary[6] on this verse attached all importance to the word *āsannah* (meaning approximately), explained its significance, advanced his own interpretation and proceeded his point why the practice of taking ($\sqrt{10}$) D as the exact measure of C was not justified. The present article will throw light

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**There are in all 16 quotations in Sanskrit namely *S -1, S - 2, ..., S - 16*, all given in Appendix on his methodology based on the then traditional Geometrical knowledge (in India) wherefrom the practice of taking $C = \sqrt{10}\ D$ originated and also on his note revealing what in modern style we express as the incommensurability of π, hinting at its non-constructible nature (a real number is said to be contructible if it can be obtained Geometrically by the use of a straight edge and compass).

INTERPRETATION OF THE WORD ĀSANNAḤ

Bhāskara (I) interprets [*S* - 2] that the word *āsannaḥ* (in the above-mentioned *Āryabhaṭīya* verse (*S* - 1) meaning 'near' (*nikaṭaḥ*) 'refers to the precise estimate and not a rough estimate for practical purpose (*sukṣmasyāsanna iti na pun arvyāvahārikayostulyā*). To emphasize the point he added [*S* - 3], 'why has an approximation (a precise estimate) of the circumference been referred to and not the exact measure of it ? It is believed that there is no method by which accurate measure of circumference (in units in which *D* is measured) can be found' (*athāsannapardhiḥ kasmāducyate, na punaḥ sphuṭaparidhirevocyate ? evaṃ manyante - sa upāya eva nāsti yena sūkṣmaparidhirānīyate'*). This interpretation when recast in the modern shape, will take the following form: *Āryabhaṭa* (I)'s above-mentioned verse provides us with a precise estimate of π and not a rough estimate of it, the question of finding accurate value of π being left unattainded as it was believed that there was no method to obtain it'.

BHĀSKARA (I)'S CRITICISM AGAINST TAKING $C = (\sqrt{10})D$

As the context to the above interpretation, Bhāskara now quotes a verse [*S* - 4] in prākṛt giving the relation $C = (\sqrt{10})\ D$ and proceeds to criticize it. He comments [*S* - 5], 'here also (there is) only an approach (to the problem) and no solution (of it)' : *'atrāpi kevala evāgamaḥ naivopapattiḥ'* or in other words, here also only an approximation (of C in terms of D and consuquently, of C/D) and no exact value of it is obtained.

INCOMMENSURAITY OF C IN TERMS OF D

To clarify the above point of disagreement, Bhāskara (I) considers a circle of unit diameter and resorts to one particular mode of proof called *'pratyakṣa'* (which is actually, the first[7] mode of proof prescribed in each of *Vedānta, Nyāya and Sāṃkhya*) to establish his views 'on the relation between C and D. This method involves in this case, direct measuring of the circumference of the circle drawn with (the standard) unit (length) as the measure of a diameter to have the ocular evidence that the circumference will always outstrip[8] — *'pratyakṣeṇaiva pramīyamāna rūpaviṛkambhakṣetrasya paridhiḥ'*; where *pramī* (= *pra* - $\sqrt{}$ *mi*) means to outstrip or to surpass and *pramīyamāna* (= *pra* - $\sqrt{}$ *mi* + *śānac* , passive) means that the incident of 'being outstripped will continue.' As nothing has been mentioned herein about the scale in which the unit of length has been chosen, it is clear that according to Bhāskara (I), the circumference will ever outstrip when measured directly in terms of a unit, however small that unit may be. Since in this case D = 1 unit (of length), C cannot be measured by the same standard scale of measurement in which D is measured or in other words, C/D is incommensurable. Bhāskara then adds that this nonavailability of exact measure of C in terms of D (= 1 unit) is not due to non-availability of exact value of a surd - *'naitat, aparibhāṣita pramāṇatvāt karaṇīnām'* (*aparibhāṣita* - not established as a rule, *pramāṇa* - correct or exact value) i.e., C/D is not a surd.

Bhāskara's Attempt To Discard $C/D = \sqrt{10}$;

Bhāskara (I) now intends to show that for a given D, C cannot be such as $C/D = (\sqrt{10})$. He considers the case from two different angles :

(i) from quadrature point of view, and

(ii) from the stand point of rectification.

Justification from Quadrature point of view

Bhāskara observes that [*S* - 6] 'a rectangle 3 unit long and one unit wide having a diagonal of length ($\sqrt{10}$) unit, can be judged as circumscribed by the circle drawn with a diameter equal to the diagonal and the area of the circle can be found mathematically'. He points out that [*S* - 7] 'a circle circumscribing a rectangle consists of four segments of the circle (lying outside the rectangle) and the rectangle circumscribed (*vṛttakṣetre catvāri dhanukṣetrāni, ekamāyatacaturaśrakṣetram*). The sum of the areas of these five components should equal the area of the circle (*teṣām phalasamāsena vṛttakṣetraphalena bhavitabyam*). But (if $\sqrt{10}$ be taken for C/D) the said equality will not hold; To substantiate this proposition he considers the following example I (*tatpratipādanārthamuddeśakaḥ*) where he uses certain terminologies which need clarification:

(a) *avagāhya* or *avagāha* or *iṣu* or *śara* : these are synonymous, to denote the length of the perpendicular at the midpoint of a chord of a circle and extended upto the nearer arc.

(b) jīva-chord of a circle

(c) *dhanukṣetra* or *dhanupaṭṭa*-segment of a circle

***Example* I** : In a circle of diameter 10 unit consider two chords, each six unit long, one in the East and the other to the West, having the respective *avagāhyas* each of unit length and two other chords, each of length eight unit, one in the sourthern part and the other in the northern part (of the circle), with respective *avagāhyas* each of length 2 unit' [*S* - 8]. According to the above description, let ABCD be a rectangle circumscribed by a circle with diameter ten unit (having *avagāhyas* EM = 1 unit = WM' and NP = 2 unit - SP')

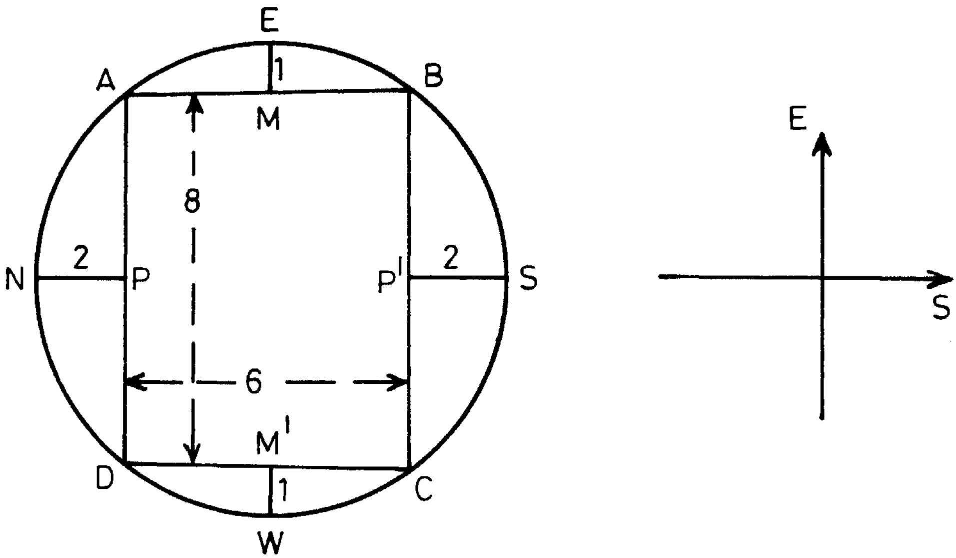

Fig. 1

Bhāskara plans for finding the areas of the following four segments of the circle:

(i) AEBA, (ii) DCWD, (iii) ADNA, (iv) BSCB

He then quotes the following three verses (in *prākṛt*) as the mathematical tools necessary for his experiment with the said example (I).

Tool (1) : Verse giving the rule for finding the length of a chord (of a circle) (*jivānāmānayanopāyasūtram gāthā*):

'The diameter less *avagāha* should be multiplied by *avagāha*. The square-root of the four times of this (product) is always the length of the (corresponding) chord' [*S* - 9]

i.e., in the above figure, AB = $\sqrt{4EM(EW— EM)} = \sqrt{4 \times 1 \times 9}$ unit

i.e., *AB* = 6

Similarly, *DC* = 6 unit *AD* = 8 unit = BC

Tool (2) : Verse on the rule for area of a segment of a circle (the arc being separated by a chord of a given *avagāhya* or *isu* or *avagāha*.

(*dhanuhkṣetraphalānayane sūtram gāthā*) :

The product of a chord and one-fourth of the (corresponding) *isu*, when multoplied by $\sqrt{10}$ should give the area of the segment of a circle' [*S* - 10]. With the help of this verse (*anayā gāthaya*), the area of each of the segments AEBA and DCWD is unit $\sqrt{10}\ .6.\frac{1}{4}\ .\ 1$ sq. unit = $\sqrt{\left(\frac{90}{4}\right)}$ sq. unit, so that the sum of the areas of these two segments

$=\sqrt{90}$ sq. unit. Also, the area of each of the segments ADNA and BSCB is $\sqrt{10}.8.(\frac{1}{4}.2)$ sq. unit $=\sqrt{160}$ sq. unit, so that the sum of the areas of these two segments= $2.\sqrt{160}$ sq. unit $=\sqrt{640}$ sq. unit.

Tool (3) : Verse on the rule for addition of two similar surds of the forms $\sqrt{10\,x^2}$ and $\sqrt{10\,y^2}$ (x, y being two rational numbers) (*karaṇiprakṣepasūtram gāthā*)

'The numbers (under the radical i.e., $10x^2$ and $10y^2$) are divided by 10 (*apavartya ca daśakena*), the sum of the square-roots (of the quotients) i.e., x + y (*mūlasamāsaḥ*) is raised to the same power (as that of the quotients (*samottham*) i.e., x + y is squared and then multiplied by the divisor (*apavartanāṅkaguṇitam*) i.e., by 10 to obtain $10\,(x+y)^2$; the square-root of which is the sum of the two (similar) surds [*S* - 11]. With the help of this verse, $\sqrt{90}+\sqrt{640}=\sqrt{10\,(3+8)^2}=\sqrt{1210}$. Next, taking the area of the rectangle ABCD, as $\sqrt{48^2}$ sq. unit i.e., as $\sqrt{2304}$ sq. unit and that of the circle (of diameter 10 unit) as $\sqrt{10}\,.\,25$ sq. unit, Bhāskara concludes that the area of the circle becomes less than the sum of the areas of the components (*karaṇisamāskriyayā samasyamāne raśyorsaṃkṣepatā*)

i.e., $\sqrt{90}+\sqrt{640}+\sqrt{2304}=11\sqrt{10}+\sqrt{2304}>25\sqrt{10}$

JUSTIFICATION FROM THE RECTIFICATION POINT OF VIEW

Bhāskara now turns towards the comparison of the length of an arc to the chord (separating the arc) of a given *iṣu* (or śara) with the commenent that 'the practice of taking $C=(\sqrt{10})\,D$ does not always provide us even with a method for finding an arc-length (*pṛṣṭhānayanamapi ca daśakaraṇīparidhiprakriyā parikalpanayā sadā na [bhavati]* and quotes the then rule for finding the length of an arc of a circle (*pṛṣṭhānayanasūtrārdham*) (to use it as a tool in the ensuing experiment)

Tool (4) : One-fourth of the length of a chord (is) added to the half of the corresponding śara and the sum (is) multiplied by itself (i.e., squared). The square root of ten times the product is the arc-length. [*S* - 12]

(*jyāpādaśarārdhayutiḥ svaguṇāḥ daśasaṅguṇā karaṇyastāḥ*)

The experiment he performs to corroborate his comment is based on the following examples :

Example II : 'In a circle of diameter 52 unit consider an arc specified by the avagāhya of 2 unit'

(*dvipañcāśadviṣkambhe dviravagāhya*)

i.e., in a circle of diameter 52 unit consider an arc separated by a chord AB of a given *śara PQ* - 2 unit.

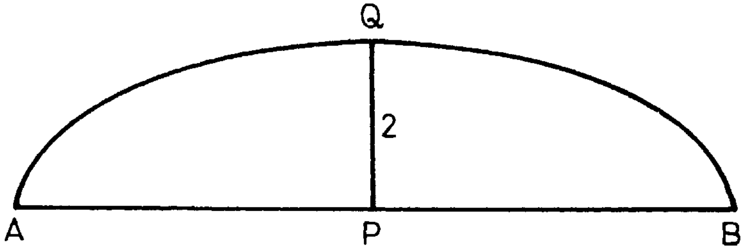

Fig. 2

He calculates :

(i) length of the chord AB = 20 unit, by the tool (1)

(ii) length of the arc AQB = $\sqrt{10}\ (\frac{AB}{4} + \frac{PQ}{2})^2$ = 360 unit and

(iii) $\overline{AB}^2$ = 400 = 400 (*sakalajyāvargaścatvāri śatāni*).

Bhāskara then, argues 'how is it possible that a chord of length $\sqrt{400}$ unit separates an arc of length $\sqrt{360}$ unit ? (as) the arc will be longer than the chord' (*jyāyasa jyātaḥ pṛṣṭhena : jyāyas-larger/longer, jyā-chord, pṛṣṭha - arc*) [*S* - 13]

Repeating his argument with the following example :

Example III : 'In a circle of diameter 26 unit, consider an arc specified by the avagāhya of one unit' (*saḍvimśativiṣkambhakṣetre ekamavagāhya*) where, following the above rule, a chord of length 10 unit separates an arc of length $\sqrt{90}$ unit, he gets at a similar conclusion.

Bhāskara's inference from the two experiments :

In consideration of observations of the above two experiments Bhāskara (I) discards the practice of taking $C = (\sqrt{10})\ D$ as erroneous (*avicārita*) and states that$(\sqrt{10})$ *D* is a very rough estimate of C (*atyantasthūlatāmāpannam*) [*S* - 14).

Discussion

The above detals of Bhāskara (I)'s investigation following his elaborate discernment about the actual situation in defiance of the erroneous practice of taking $\sqrt{10}$ for the value of (*C/D*) has two aspects namely:

Aspect (A) The scientific bias underlying his attempt to establish that 'for no circle C/D is exactly equal to $\sqrt{10}$ has been prominent in his processes of investigation. The examples set by him aim at solving the problem from (i) the quadrature-point of view and (ii) the stand point of rectification. To judge the case from the angle of quadrature he followed a method comparable to the method of exhaustion (with the help of Geometrical rules at his disposal) showing a wide departure from that of Eudoxus.[9] Of the four tools, those embodying the rules: (a) on the area of a circular segment (available in *Bṛhatkṣetrasamāsa of Jinabhadragani* - a Jaina Mathematician during the period 529-589 AD)[10] and (b) on the length of a circular arc, both suffer from unreliability in the sense that they might have been framed by analogy[11] from the respective rules applicable to a semicircle, while the other two rules namely:

(c) the rule for finding the length of a chord interms of segments of te diameter bisecting it (known to Indian Mathematicians since the times of writing *Sūryaprajñapti*[12] in about 400BC[13]) can be deduced from the propositions 3 and 35 jointly of the Book III in the *Euclid's Elements.*[14]

(d) the rule for addition of two similar surds of the forms $\sqrt{xy^2}$ and $\sqrt{xz^2}$, (where y, z are non-zero rational numbers and x is a non-zero, non-square rational number) with the incidental mentioning of the art of establishing order relation between two similar surds of the forms $\sqrt{x}$ and $\sqrt{y}$ (x, y are non-square rational numbers) by comparing their squares, are undoubtedly instances of Indian Mathematical heritage.

Aspect(B) Ocular evidence of actual occurence of an event *i.e.* drawing of inference of the basis of actual observation (made in a practical experiment), called *'pratyakṣa'* was the way Bhāskara followed to get at the truth that 'C always outstrips when measured by a unit length (which might have been chosen as small as the system of linear measurement provided in this times). Bhāskara's views at this point gained ground much later through Mādhavācārya's (1340-1425AD) Polygonal Approximation to Circle[15] and was clarified by Nīlkaṇṭha Somayāji[16] (1443-1543 AD) in his following observation [*S* - 15].

'If the quantity by which the diameter is gradually made to diminish without remainder, be taken to diminish the circumference step by step there will be a remainder (i.e. if the diameter measured by a unit of length be commensurable with respect to the unit, the circumference will be incommensurable) and if the quantity by which the circuference is made to diminish without remainder be taken to diminish the diameter gradually, there will be left a remainder (*i.e.*, if the circumference be commensurable with respect to a unit, the diameter will not be so). Thus both (of them) are not commensurable with respect to the same unit', √mī— to lessen, *mīyamānaḥ* - getting diminished step by step, *niravayavaḥ* - without remainder) speaking of the irrationality of C/D. This nature of C/D ($=\pi$) was confirmed by Śaṅkara Pārāśava[17] (1500-1560 AD) thus:

'thus even by computing the results progressively it is impossible logically to come to the finality' (*evaṃ muhuḥ phalānayne kṛte'pi yuktitaḥ kvāpi na samāptiḥ* [*S* - 16]).

Inspite of the fact that the incommensurability of C/D (= π) was just conjectured by Bhāskara (I), it was established much later by Mādhavācārya and thereafter confired by Śaṅkara Pāraśava, the clarity of Bhāskara's conjecture demands special credit as it points out two things namely,

(1) C/D (*i.e.*, π) is incommensurable with respect to a unit of linear measure (*pratykṣeṇaiva pramīyamāṇo rūpaviṣkambhakṣetrasya paridhiḥ* (*i.e.*, if D = a unit length in a scale of linear measure and if C be measured by the unit then C will always outstrip).

(2) C/D (*i.e.*, π) is not a surd (*naitat aparibhāṣitapramāṇatvāt karaṇīnām i.e.* this incommensurability is not due to the non-availability of the exact value (as a rational number) of a surd.

Bhāskara's conjecture therefore, states that for a circle of unit diameter, the length of the circumferece being neither a surd nor measurable in terms of the unit lenth, it is impossible to construct by using ruler and compass (*i.e.*, by Euclidean methods) a length equal to the circumference and this justifies the remark, *Sa upāya eva nāsti yena sūkṣmaparidhirānīyate' i.e.*, in the relation $C = KD$, k (*i.e.*, π) is not constructible (or Euclidean). Herein was latent the modern concept that π is transcendental - a truth attained by F Lindemann[18,19] (1882 AD) about 1300 years after Bhāskara I (574 AD)

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APPENDIX

S-1: चतुरधिकं शतमष्टगुणं द्वाषष्टिस्तथा सहस्राणाम् ।
अयुतद्वयविष्कम्भस्यासन्नो वृत्त परिणाहः ॥ १० ॥
(caturadhikaṃ śatamaṣṭaguṇaṃ dvaṣaṣṭistathā sahasrāṇām ayutadvayaviṣkambhasyāsanno vṛtta pariṇāhaḥ //)

S-2: आसन्नः निकटः । कस्यासन्नः ? सूक्ष्मस्य परिणाहस्य ।
कथं विज्ञायते सूक्ष्मस्यासन्न इति, न पुनर्व्यावहारिकस्यासन्नः ।
यावता श्रुतपरिकल्पना सूक्ष्मव्यावहारिकयोस्तुल्या ।
āsannaḥ nikaṭah kasyāsannaḥ ? sūksmasya pariṇāhasya / katham vijiñāyate sūksmasyāsanna iti, na punarvyāvahārikasyāsannaḥ / yavatā śrutaparikalpanā sūksmavyāvahārikayostulyā /)

S-3: अथासन्नपरिधिः कस्मादुच्यते, न पुनः स्फुटपरिधिरेवोच्यते ?
एवं मन्यते — स उपाय एव नास्ति येन सूक्ष्मपरिधिरानीयते ।
(athāsannaparidhiḥ kasmāducyate, na punaḥ sphuṭaparidhirevocyate ? evam manyate —sa upāya eva nāsti yena sūksmaparidhirāniyate /)

S-4: विक्खंभवग्गदसगुणकरणी वट्टस्स परिरओ होदि । [Ref. p. 72]
(विष्कम्भवर्गदशगुणकरणी वृत्तस्य परिणाहो भवति ।
(vikhambhavaggadaśaguṇakarṇī vaṭṭassa parirayo hodi)
[viṣkambhavargadaśaguṇakaraṇī vṛttasya parināho bhavati]

S-5: अत्रापि केवल एवागमः नैवोपपत्तिः । रूपविष्कम्भस्य
दशकरण्यः परिधिरिति । अथ मन्यते प्रत्यक्षेणैव
प्रमीयमाणो रूपविष्कम्भक्षेत्रस्य परिधिर्दशकरण्य इति ।
नैतत्, अपरिभाषितप्रमाणत्वात् करणीनाम् ।
(atrāpi kevala evāgamaḥ naivopapattiḥ / rūpaviṣkambhasya daśakaraṇyāḥ paridhiriti / atha manyate pratyakṣenaiva pramīyamāno rūpaviṣkambhakṣetrasya paridhirdaśakaraṇya iti / naitat, aparibhāṣitapramānatvāt karaṇīnām.

S-6: एकत्रिविस्तारायामायतचतुरश्रक्षेत्रकर्णेन दशकरणिकेनैव तद्विष्किम्भपरिधिर्व्येष्टयमाणः
स तत्प्रमाणो भवतीति चेत्रदपि साध्यमेव ॥
(ekatrivistārāyāmāyatacaturaśrakṣetrakarṇenadaśakaraṇīkenaiva tadviṣkambhaparidhirbeṣṭyamānaḥ sa tatpramano bhavatīti cettadapi sādhyameva)

S-7: वृत्तक्षेत्रे चत्वारि धनुक्षेत्राणि, एकमायतचतुरश्रक्षेत्रम् ।
तेषां फलसमासेन वृत्तक्षेत्रफलेन भवितव्यम् ।
तानि फलानि संयोज्यमानानि न वृत्तक्षेत्रफलतुल्यानि भवन्ति ।
(vṛttakṣetre catvāri dhanukṣetrāni ekamāyatacaturaśrakṣetram / teṣām phalasamāsena vṛttakṣetraphalena bhavitavyam / tāni phalāni saṃyojyamānāni na vṛttakṣetraphalatulyāni bhavanti)

S-8: दशविष्कम्भक्षेत्रे पूर्वापरभागे एकरुपमवगाह्य ।
जीवा षड्, दक्षिणोत्तरयोरपि द्वे रूपेऽवगाह्याष्टौ ॥
(daśaviṣkambhakṣetre purvaparabhāge ekarūpamavagāhya / jība sad, dakṣiṇottarayorapi dve rūpe'vagāhyasṭau /)

S-9: जीवानामानयनोपायसूत्रं गाथा —
ओदाहूणं विक्खम्भं एगाहेण संगुणं कुर्यात् ।
चउगुणिअस्स तु मूलं जीवा सव्वखत्ताणाम् ।
(अवगाहोनं विष्कम्भमवगाहेण सङ्गुणं कुर्यात् ।
चतुर्गुणितस्य तु मूलं सा जीवा सर्वक्षेत्राणाम् ॥)
(jivānāmānayanasūtram gātha -
ogāhuṇam vikkhamvam egāheṇa saṃguṇaṃ kuryāt /
cauguṇiassa tu mūlaṃ jivā savvakhattāṇām //)
[avagāhonam viṣkambhamavagaheṇa saṅguṇam kuryāt / caturguṇitasya tu mūlam sā jivā sarvakṣetrānām]

S-10: धनु:क्षेत्रफलानयने सूत्रं गाथा —
इसुपायगुणा जीवा दसकरणी भवेद् विगणिय पदम् ।
धनुपट्ट अम्मिखत्ते एदं करणं तु णाअव्वम् ॥
(इषुपादगुणा जीवा दशकरणीभिर्भवेद् विगुण्य फलम् ।
धनुपट्टेऽस्मिन् क्षेत्रे एतत्करणं तु ज्ञातव्यम् ॥)
(isupāyaguṇa jīva daśakaraṇī bhaved vigaṇiya padam I dhanupaṭṭa ammikhatte edaṃ karaṇam tu nāavvam.)
[iṣupādaguṇa jīva daśakaraṇībhirbhaved viguṇya phalam / dhanupaṭṭe'smin kṣetre tu jñātavyam /]

S-11: करणीप्रक्षेपसूत्रं गाथा —
औवट्टि अ दस्सकेण इ मूलसमासस्समोत्थवत्
ओवट्टणायगुणियं करणीद्वमासं तु णाअव्वम् ।
(अपवर्त्य च दशकेन हि मूलसमासः समोत्थ यत् ।
अपवर्तनाङ्कगुणितं करणीसमासं तु ज्ञातव्यम् ॥)

(auvaṭṭi a dassakeṇa imūlasamāssamotthavat ovaṭṭaṇāyaguṇiyaṃ karaṇīsamāsaṃ tu nāvyam /)
[apavarta ca daśkena hi mūlasamāsḥ samttha vat apavartanāṅka guṇitam karaṇīsamāsaṃ tu jñātavyam]

S-12: ज्यापादशरार्धयुतिः स्वगुणा (दशसङ्गुणा करण्यस्ताः)
(jyāpadaśarārdhayutiḥ svagunāh) [daśasaṅguṇā karaṇyastāḥ]

S-13: सकलज्यावर्गश्चत्वारि शतानि, पृष्ठं करणीनां षष्टिशतत्रयमिति,
कथमेतत् संघटते । ज्यायसा ज्यातः पृष्ठेन भवितव्यम् ॥
(sakalajyāvargaścatvāri śatani, pṛṣṭhaṃ karaṇīnām ṣaṣṭiśatatrayamiti, kathametat saṃghatate / jyāyasā jyātaḥ pṛṣṭhena bhavitavyam)

S-14: एवमिदमालोच्यमानमत्यन्तस्थूलतामापन्नमिति ।
तस्मात् स उपाय एव नास्तीति सुक्तम् । [Ref. p. 75]
(evamidamālocyamānamatyantasthūlatāmāpannamiti /
tasmāt sa upāya eva nastīti suktam)

S-15: येन मानेन मीयमानो व्यासो निरवयवः स्यात
तेनैव मीयमानः परिधिः पुनः सावयव एव
स्यात । येन च मीयमानः परिधिर्निरवयवस्तेनैव
मीयमानो व्यासोऽपि सावयव एव इत्येकेनैव
मानेन मीयमानेनरूभयोः क्वापि निरवयवत्वं स्यात् ।
(yena mānena mīyamāno vyāso niravayavaḥ syāt
tenaiva mīyamānaḥ paridhiḥ punaḥ sāvayava eva syāt /
yena ca mīyamānaḥ paridhirniravayavastenaiva mīyamāno vyāso'pi sāvayava
eva ityekenaiva mānena mīyamānayorubhayoḥ kvāpi niravayavatvaṃ syāt /)

S-16: एवं मुहुः फलानयने कृतेऽपि युक्तितः क्वापि न समाप्तिः ।
(evam muhuḥ phalānayane kṛte'pi yuktitaḥ kvāpi na samaptiḥ //)