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GOVINDASVĀMIN'S ARITHMETIC RULES CITED IN THE KRIYĀKRAMAKARĪ OF ŚAṄKARA AND NĀRĀYAṆA

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Govindasvāmin's arithmetic rules are preserved as quotations in the *Kriyākramakarī* commentary by Śaṅkara and Nārāyaṇa on Bhāskara II's *Līlāvatī*. These rules are collected at one place in this paper and provided with literal English translation and mathematical comments. One of the notable features of Govindasvāmin's arithmetic is his elaborate treatment of the three-quantity operation (*trairāśika*, usually rendered as Rule of Three) and its comparison to the inference (*anumāna*) of Indian logic.

Keywords: arithmetical operations, Govindasvāmin, medieval Indian mathematics, Rule of Three.

0. INTRODUCTION

Govindasvāmin worte at least five treatises in the field of *jyotiṣa*, that is, two in astronomy, *Govindakṛti* and *Mahābhāskarīyabhāṣya* (commentary on Bhāskara I's *Mahābhāskarīya*), two in horoscopic astrology, *Govindapaddhati* and *Prakaṭārthadīpikā* or *Saṃpradāyadīpikā* (commentary on Parāśara's *Horāśāstra*), and one in mathematics, *Gaṇitamukha*.[1] Out of these, only the two commentaries are extant and the rest, which are his original works,

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[1] Kuppanna Sastri 1957, xlvi-xlix; Raja 1963, 127-128; Sarma 1972, 44-45; CESS A2, 143b-144a; A3, 35b; A4, 86b, 101a.

are known only fragmentarily through quotations by Kerala mathematicians such as Śaṅkaranārāyaṇa, Nīlakaṇṭha, Śaṅkara Vāriyar, etc.

The date of Govindasvāmin has been estimated to be *ca.* AD 800-850 because Nīlakaṇṭha refers to the tradition that Śaṅkaranārāyaṇa, an astronomer at the court of the Kerala King Ravivarma at Mahodayapura, was a pupil (*śiṣya*) of Govindasvāmin and because Śaṅkaranārāyaṇa refers to the Śaka year 791= AD 869 in his commentary on Bhāskara I's *Laghubhāskarīya.*[2] Presumably Govindasvāmin, too, was from Kerala because most of those who quote from his works were from there and because the manuscripts of his extant works are confined to south India.

Govinda's contributions in trigonometry (especially concerning a sine table), attested in the *Mahābhāskarīyabhāṣya*, have been commented on by Gupta (1969, 91-92 and 1971), and his twenty-two Āryā stanzas for *kuṭṭākāra* ("pulverizer") or a solution of linear indeterminate equations, which Śaṅkaranārāyaṇa cites from the *Govindakṛti* in his commentary on the *Laghubhāskarīya*, have been translated into English by Shukla (1963, 103-114). The present paper aims at providing Govinda's arithmetic rules in Sanskrit collected from the *Kriyākramakarī* together with my English translations and comments.

The *Kriyākramakarī*, written by Śaṅkara and Nārāyaṇa, is a commentary on Bhāskara II's famous textbook on arithmetic and mensuration, *Līlāvatī* (AD 1150). Śaṅkara Vāriyar wrote his commentary up to Stanza 199 of the *Līlāvatī* by about AD 1540,[3] when he stopped writing "due to many kinds of works" in which he was involved, and Nārāyaṇa of Mahiṣamaṅgala, son of another Śaṅkara, began to write the rest of the commentary when he was eitghteen years old, perhaps about AD 1560, after the death of

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[2] Kuppanna Sastri 1957, xlvii. Shukla, 1976, lxxxviii-lxxxix, points out the possibility that Govindasvāmin was either anterior to or a senior contemporary of Haridatta (fl. AD 683).

[3] Under the rule of inverse operations (p. 98), Śaṅkara gives an example of a calculation of the *ahargaṇa* or the number of the days elapsed since the beginning of the present Kaliyuga, i.e., 18 February 3102 BC. The answer obtained is 1692972 days, which correspond to a day in AD 1534. See Sarma's Introduction, pp. xxi-xxii.

the former.[4]

The *Kriyākramakarī* contains a number of quotations from other works. In the words of the editor, "His (Śaṅkara's) contributions are remarkable, among other things, for the wealth of quotations they preserve from ancient authorities like Govindasvāmin, Śrīdhara and Jayadeva."[5]

Thirty-three passages in total have been cited from Govindasvāmin's works in Śaṅkara's, part of the *Kriyākramakarī*, while one in Nārāyaṇa's. Five of the passages citied by Śaṅkara concern astronomy and the rest deal with arithmetic. The sources of all astronomical passages, which Śaṅkara cites as examples of arthmetic computations, are known: two Āryā stanzas from the *Govindakṛti*[6] and three prose passages from the *Mahābhāskarīyabhāṣya*.[7] Only two of the arithmetical passages can be

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[4] *Nārāyaṇa* writes at the beginning of his part (p. 391): *itīdaṃ gaṇitavidagresareṇa śrīhutāśākhyadevālayaparicārakena śaṅkarapārāśavena vyākhyātam / tasya bahuvidhavyāpārapāratantryāt tatra vyāpāraś ca nivṛttaḥ / tasmin svargate punar mayā puruvanagrāmajena vipreṇa gṛhanāmnā mahiṣamaṅgalena śaṅkarātmajena nijanāmnā nārāyaṇenāṣṭādaśavayaskena śiṣyaprārthanayā tatpitṛniyogena ca yathākathaṃcid eva vyākhyānam ārabdham //* ("This much has been explained by Śaṅkara Pārāśava (= Vāriyar), a leading figure among mathematicians, who was a functionary of a temple called Hutāśa ('fire'). Due to his engagement in many kinds of works, his engagement in this <commentary> was prevented. When he went to heavens, on the other hand, a commentary <in succesion to my predecessor's> was, at the request of pupils and under the direction of my (lit.his) father, undertaken by me, son of Śaṅkara, a brāhmaṇa of eighteen years old, born at Puruvanagrāma, Nārāyaṅa by name, whose family name is Mahiṣamaṅgala.") Śaṅkara's works méntioned here seem to include his *Karaṇasāra*, the epoch of which falls in AD 1554, and his commentary, *Laghuvivṛti*, on Nīlakaṇtha's *Tantrasaṃgraha*, which is said to have been written in AD 1556 (Sarma 1975, xvii; 1977, lxi-lxii). Notation in my translation of Sanskrit passages: A pair of angled brackets, <A>, indicates that A is not a translation of Sanskrit word(s) but has been supplied by me. A pair of square brackets, [A], indicates that A is a number expressed in the so-called word numerals (also called *bhūta-saṃkhyā*).

[5] Sarma's Introduction to his edition, p. xxii.

[6] One stanza, cited in SL p. 186, lines 8-9, is on the mean longitude of a planet, and the other, cited in SL p. 190, lines 22-23, on the refined anomaly of a planet.

[7] The first passage cited in SL p. 98, lines 24-25 is on the remainder of rotation of the sun (GMB 1.47, p. 61, lines 6-7), the second in SL p. 186, lines 1-4 on the mean longitude of a planet (GMB 1.9, p. 13, lines 19-22), and the third in SL p. 190, lines 19-21 on the refined anomaly of a planet (GMB 4.21, p. 197, lines 4-7).

identified: one Āryā stanza on the addition and subtraction of fractions from the *Gaṇitamukha* (see § 2.4 below) and a prose passage on the three-quantity operation (*trairāśika*, usually rendered as "rule of three") from the *Mahābhāskarīyabhāṣya* (see § 3.1 below). The remaining passages on arithmetic rules, all composed in the Āryā measure, may also have been cited from the *Gaṇitamukha* ("Introduction to Mathematics"). The only quotation by Nārāyaṇa[8] is a long prose passage from the *Mahābhāskarīyabhāṣya*.[9] It gives two interpretations of the word *kuṭṭākāra*[10] and definitions of the two kinds of *kuṭṭākāra*, "residual" (*sāgra*) and "non-residual" (*niragra*).[11]

I have rearranged the arithmetic rules cited by Śaṅkara in three sections, that is, 1. basic operations of integers, 2. basic operations of fractions, and 3. the three-quantity operation. I have also included a supplementary rule composed in four Anuṣṭubh stanzas for the three-quantity operation (see § 3.5 below), which is found in Govinda's *Mahābhāskarīyabhāṣya*, and a rule in one Anuṣṭubh stanza for the double three-quantity operation (see § 3.7 below), which Nīlakaṇṭha ascribes to Govinda in his commentary on the *Āryabhaṭīya*.

1. Basic Operations of Integers

1.1 Division

Govinda's rule for division itself is not known but a verse of his is cited by Śaṅkara for the cancellation of a common factor of the divisor and dividend.

govindasvāmināpy uktam —

bhājyāṃśāṃś cānyau vā chindyād anyonyabhaktaśeṣeṇa /

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[8] NL p. 438, lines 4-19.

[9] GMB 1.41, p. 54, line 8 – p. 55, line 7.

[10] First interpretation: *kuṭṭākāra* is analyzed as *kuṭṭā* + *kāra* and means "one which causes a special kind of division (*viśiṣṭa-cchedana*)," that is, the multiplier, x, in the equation, $y = (ax + c)/b$. Second interpretation: *kuṭṭākāra* is analyzed as *kuṭṭa* + *ākāra*, where *kuṭṭa* is equated with *kuṭṭākāra* in the first sense, and means "the calculation (*gaṇita*) which produces the *kuṭṭākāra* in the first sense (x)."

[11] The former treats the problem, $N = a_i x_i + R_i\ (0 \leq R_i < a_i)$, and the latter, $y = (ax + c)/b$.

tatrāptau tāv eva dṛḍhāv idam apavartanaṃ karma // iti / [12]

"It has been said by Govindavāmin, too —

"One should divide dividends (*bhājya*) and numerators (*aṃśa*), or any other pair <of numbers>, by the <last> remainder <obtained when they are> mutually divided. The two obtained there are firm (*dṛḍha*) (i.e.mutully prime). This is a computation of reduction (*apavartana*)."

The mutual division mentioned here is the so-called Euclidean algorithm employed for obtaining the greatest common factor of two numbers. The word, *bhājya*, ("dividend") at the beginning of the first line is out of place. This verse is identical with the one cited in the section for fractions (SL p. 61, lines 7-8) except for the word in question, that is, *bhājya*, for which the latter reads *cheda* (denominator), which fits the context. See § 2.3 below. Govinda gives a similar rule as a part of his rules for *kuṭṭākāra* in his *Govindakṛti*, where it is meant for the cancellation of the greatest common factor of a and b in $y = (ax + c) / b$.

guṇakārabhāgahārau vibhajed anyonyabhaktaśeṣeṇa /

tau tatra bhājyahārau dṛḍhāv avāptau vinirdiṣṭau // 3 // [13]

"One should divide the multiplier and the divisor by the <last> remainder <obtained when they are> mutually divided. The dividend and divisor obtained there are said to be firm."

Here also, there seems to be some confusion of terminology. The a in $y = (ax + c) / b$ is called "multiplier" (*guṇakāra*) in the first compound of the first line but "dividend" (*bhājya*) in the second line: the former is unusual and the latter is traditional, although it could logically be regarded as "a multiplier" of x.

For the use of the Euclidean algorithm in *kuṭṭākāra*, see AB 2.32, MB 1.41, BSS 18.3 + 9, GSS 6.115 + 136, MS 18.1, SS 14.27, L 243, BG 51, BA 1.54.

For division, see PG 22 = Tr 9, GP 11, GSS 2.18-19, MS 15.4, GT 18 (p. 6, lines 21-24), SS 13.3, MU 2.2.109-112, L 18, GK 1.16, GS 1.33, PV X10, GM 19+21.

[12] SL p. 20, lines 11-12.

[13] Shukla 1963, 104.

1.2 Calculation of the Square

Govinda's rule for calculating the square of an integer expressed in the decimal place-value notation is the one commonly met with.

tad uktaṃ govindasvāminā —

athavopary antyapadaṃ[14] *svahataṃ vinidhārya tatpadaṃ dviguṇam /*

ādipadopari nihitaṃ[15] *śeṣapadair āhataṃ kṛtvā //*

utsāryotsāryaitad athavāpasāryāpasārya śeṣapadam /

śeṣapade karmaivaṃ kartavyaṃ tatra vargāptiḥ // iti / [16]

"It has been said by Govindasvāmin—

"Or, otherwise, one should put the latter term multiplied by itself above <the latter term>, multiply that <latter> term multiplied by two, placed above the former term(s), by the remaining terms, // shift either this <result> upward or the remaing terms downward, and perform the same computation with regard to the remaining terms. Then one obtains the square."

This rule is illustrated in Table 1, where *a* is the "latter" term and *b* and *c* combined are the "former" terms in the first step.

For the calculation of the square, see AB 2.3, BSS 12.62, PG 23-24= Tr 10-11, GP 11, GSS 2.29+31, MS 15.6, GT 20 (p. 7, lines 24-25)-21, SS 13.4, L 19, GK 1.17, GS 1.34, PV X11, GM 23.

1.3 Extraction of the Square-Root

Govinda's rule for extracting the square-root of an integer expressed in the decimal place-value notation is the one commonly met with.

govindasvāmināpy uktam —

ṛṇam antyād viṣamapadāt kṛteḥ kṛtir yasya hīyate tena /

dviguṇenānantarato labdhaṃ nyasya tadanantarataḥ / /

tadvargam uparirāśes tyaktvā dvitāditaṃ tac ca /

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[14] *antapadaṃ* SL/AB. The abbreviations, SL/A, SL/B, etc. denote the mss. used by K. V. Sarma for his edition of the SL.

[15] *nihataṃ* SL/CD.

[16] SL p. 24, lines 4-7.

tena punas sarveṇa tathānte mūlaṃ dviguṇadalam[17] / / *iti* / [18]

"It has been said by Govindasvāmin, too —

"From the last odd term of a square number, the square of a certain <greatest possible number> is subtracted, and when one has put down the quotient of <the division of> the next place by twice that <number> in the next place, // and subtracted the square of it (the quotient) from the above, it (the quotient), too, is doubled. Again, by the entire <line of the doubled numbers moved to the next place, division is made> in the same manner. In the end, half of the doubled <numbers> is the square-root."

Table 1: Calculation of the square of a three-digit number, $(a.10^2+b.10+c)$.

Put the square a^2	a^2				
above a:	a	b	c		
Put the products of $2a$	a^2	$2ab$	$2ac$		
and the rest above each:	a	b	c		
Delete a and move	a^2	$2ab$	$2ac$		
the rest to the right :		b	c		
Put the square b^2	a^2	$2ab$	b^2+2ac		
above b :			b	c	
Put the product of $2b$	a^2	$2ab$	b^2+2ac	$2bc$	
and the rest above it:			b	c	
Delete b and move	a^2	$2ab$	b^2+2ac	$2bc$	
the rest to the right:				c	
Put the square c^2	a^2	$2ab$	b^2+2ac	$2bc$	c^2
above c :					c

This rule in illustrated in Table 2, where the square-root of the number obtained in Table1 is to be obtained. The role of the first word, *ṛṇa*, of the quoted verse, which usually means "a debt" or "a negative quantity", is not known.

[17] *dviguṇatalam* SL/A, *dviguṇataphalam* SL/B, *dviguṇitadalam* SL/D.
[18] SL p. 40, lines 13-16.

Table 2: Extraction of the square-root.

	10^4	10^3	10^2	10^1	10^0
	a^2	$2ab$	b^2+2ac	$2bc$	c^2
Subtract a^2 from [10^4]:		$2ab$	b^2+2ac	$2bc$	c^2
put down $2a$ in [10^3]:		$2a$			
Divide [10^3] by $2a$:			b^2+2ac	$2bc$	c^2
Put down the quotient b in[10^2]:		$2a$	b		
Subtract b^2 from [10^2]:			$2ac$	$2bc$	c^2
Double the b :		$2a$	$2b$		
			$2ac$	$2bc$	c^2
Move the line:			$2a$	$2b$	
Divide [10^1]by ($2a$ + $2b$):					c^2
Put down the quotient c in [10^0]:			$2a$	$2b$	c
Subtract c^2 from [10^0]:					
Double the c :			$2a$	$2b$	$2c$
Halve the line:			a	b	c

For the square-root, see AB 2.4, PG 25-26 =Tr 12-13, GP 12, GSS 2.36, MS 15.6-7, GT 23= SS 13.5, L 22, GK 1.19-20, GS 1.37-38, PV X12-13, GM 25.

1.4 Definition and Calculation of the Cube

Govinda's rule for calculating the cube of an integer seems slightly different from other's.

govindasvāmināpy etat spaṣṭam evoktam—
sadṛśadvādaśarāśer aśritrayasaṃhatiḥ phalaṃ sa ghanaḥ /
antyapadaṃ ātmakṛtihataṃ asyaivopari nidhāya ghanaṃ antyam[19]*//*
antyapadakṛtihatatrikaguṇitaṃ[20] *tadanantaraṃ ca padam ekam /*
apasārya tatkṛtim api tripūrvaguṇitāṃ[21] *ca nītvādhaḥ //*
taddhanam apy upayuktapadakṛtiguṇatrihatam apy upetaṃ[22] *ca /*
ghanaṃ ityādi prāgvat kuryād ghanakarma sarvapadaiḥ //iti /[23]

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19 *antyaḥ* SL/AB.

20 *antyapadāṅkṛtihatatriguṇitaṃ* SL/AB.

21 *tāṃ* .... (to the end of the quotation) omits SL/D.

22 *apetaṃ* SL.

23 SL p. 45, lines. 16-21.

"This has been told very clearly by Govindsvāmin, too—

"The product of three edges, <orthogonal to each other>, of <a solid> that has twelve equal quantities <for its edges>, is <its> fruit (i.e. the volume) <and> it is a cube. Having placed the last term multiplied by the square of itself, i.e. the cube of the last (lit. the last cube), above itself // and the next term multiplied by three multiplied by the square of the last term <above itself>, having moved downward by one <notational place>, <having placed> its square, too, multiplied by three and by the <terms in> front (on the right) <one by one above each>, having brought down,// and also having added that quantity multiplied by three multiplied by the square of the combined terms <on the right>, one should perform as before the computation of the cube beginning with the cube <of the last term> with every term <of the original number>."

Table 3: Calculation of the cube of a three-digit number, ($a.\ 10^2 + b.\ 10 + c$).

	10^6	10^5	10^4	10^3	10^2	10^1	10^0
$a^3.10^6$	a^3						
$3a^2\ (b.10 + c).\ 10^4$		$3a^2b$	$3a^2c$				
$3a\ (b.10 + c)^2\ .\ 10^2$			$3ab^2$	$6abc$	$3ac^2$		
$(b.10 + c)^3\ .\ 10^0$				b^3	$3b^2c$	$3bc^2$	c^3

Table 4: Calculation of the cube of a three-digit number according to others.

	10^6	10^5	10^4	10^3	10^2	10^1	10^0
$(a.10 + c)^3.\ 10^3$	a^3	$3a^2b$	$3ab^2$	b^3			
$3(a.10 + b)^2.\ 10^2$			$3a^2c$	$6abc$	$3b^2c$		
$3(a.10 + b)c^2\ .\ 10^1$					$3ac^2$	$3bc^2$	
$c^3\ .\ 10^0$							c^3

The first sentence defines the cube both as a number and as a geometric figure, and the rest prescribes an algorithm for performing a calculation of the cube of a number expressed in a place-value system. The details of the rule, especially concerning the shift of the original number, are not certain. Table 3 shows only the principle of the calculation of $(a.10^2 + b.\ 10 + c)^3$ according to Govinda's rule, that is,

$(a.10^2 + b.\ 10 + c)^3 = \{a.\ 10^2 + (b.\ 10 + c)\}^3$

$= a^3.\ 10^6 + 3a^2(b.\ 10 + c).\ 10^4 + 3a\ (b.\ 10 + c)^2.\ 10^2 + (b.\ 10 + c)^3.$

Table 4, on the other hand, shows the principle of the rule prescribed for the same purpose by Brahmagupta,[24] Śrīdhara,[25] and Bhāskara II,[26] which is different from Govinda's in grouping notational places.

$(a.\ 10^2 + b.\ 10 + c)^3 = \{(a.\ 10 + b).\ 10 + c)\}^3$

$= (a.\ 10 + b)^3 .10^3 + 3\ (a.\ 10 + b)^2\ c\ .10^2\ + 3(a.\ 10 + b)\ c^2.\ 10 + c^3.$

For the calculation of the cube, see AB 2.3, BSS 12.6 + 62, PG 27-28 = Tr 14-15, GP 11, GSS 2.47, MS 15.6, GT 25-26, SS 13.4, L 24-25, GK 1.21-22, GS 1.39-41, PV X11, GM 27-28.

1.5 Extraction of the Cube-Root

Govinda's verse for the extraction of the cube-root citied by Śaṅkara is incomplete. It seems to consist of the first and the last lines of the original few stanzas.

govindasvāminápi—

ghana eko dvāv aghanau punar apy evaṃ prakalpyate sthānam[27] /

āvṛtte 'smin karmaṇi ghanamūlaṃ labhyate ghanataḥ // iti /[28]

"By Govindasvāmin, too—

"One cubic and two non-cubic places are determined alternately <in the place-value expression of a number>. When this computation is repeated, the cubic root is obtained from a cube."

For the extraction of the cube-root, see AB 2.5, BSS 12.7, PG 29-31 =Tr 16-18, GP 13-14, GSS 2.53-54, MS 15.9-10, GT 29-30 (p. 13, lines 18-25) = SS 13.6-7, L 28-29, GK 1.24-25, GS 1.43-45, GM 31.

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[24] BSS 12.6.

[25] PG 27-28 = Tr 14-15.

[26] L 24-25.

[27] *sthānā* SL/A.

[28] SL p. 57, lines. 5-6.

2. Basic Operations of Fractions

2.1 Definition of Fractional Part (Numerator) and Denominator

tad uktaṃ govindasvāminā—

aṃśo rūpāvayavaḥ sa cchedo yena cchidyate[29] *rūpam /iti /*[30]

"It has been said by Govindasvāmin—

"A fractional part (numerator) is <the number of> parts of unity, <and its> divisor (denominator) is that by which unity is divided."

That is to say, in modern notation,

$$\frac{a}{p} = a.\frac{1}{p}$$

Govinda uses the words, *aṃśa* (lit. a part) and *cheda* (lit. a divisor), for the numerator and denominator of a fraction, respectively. He also uses *bhāga* (lit. a part) in the former sense though it is rare (see the next section). I will translate *aṃśa* and *bhāga* as "a fractional part"and *cheda* as "a denominator". The world *rūpa* (lit. form or colour, an object of the sense of sight) is, as usual, used to denote "unity". By extension, it also means a group of unities or an integer.

For a definition of a whole number accompanied by fractional part, see MU 2.2.119.

2.2 Positioning of Fractional Parts and Denominators

tayor nyāsasthānam api tenaivoktam—

rūpasyādhaḥsthānād aṃśasthānaṃ prakalpyate sadbhiḥ /

chedasthānaṃ[31] *tadadhas tadbhāgacchedayor evam //iti /*[32]

"The place for setting down the two (numérator and denominator), too, has been told by him (Govindasvāmin[33])—

"The position of a fractional part is set below the position of unity

[29] *yena vicchidyate* SL/AB.

[30] SL p. 61. line 19.

[31] *chedasthānāṃ* SL/AC.

[32] SL p. 61, lines 21-22. The same verse is cited also, fully or partially, in SL p. 71, lines 12-13, p. 72, line 7, and p. 73, line1.

[33] This quotation immediately follows the previous one (§ 2.1).

(integer), and the position of <its> denominator below it, by authorities. Likewise, <the position> of its fractional part and denominator <is set further below them>."

That is to say, according to Govinda, an integer accompanied by n fractional parts (a_i), the i-th denominator of which is p_i times the previous one,

$$a_0 + \frac{a_1}{p_1} + \frac{a_2}{p_1 p_2} + \cdots + \frac{a_n}{p_1 p_2 \cdots p_n},$$

is expressed as:

a_0
a_1
p_1
a_2
p_2
.
.
.
a_n
p_n

The same notation is actually employed many times in the *Bakhshālī Manuscript* for expressing a quantity in a "chain" (*vallī*) of measures. For example, the length of time, "25 years 5 months and 20 days", is expressed as:[34]

va	25
mā	5
	12
di	20
	30

where *va, mā* and *di* stand for *varṣa* (year), *māsa* (month), and *dina* or *divasa* (day) in order, and the denominators, 12 and 30, are conversion ratios: 12 months = 1 year and 30 days = 1 month. The longest "chain" in the same work consists of nine weight measures.[35] Cf. also BM Q6, PG 41

[34]Hayashi 1995, 248.
[35]Hayashi 1995, 250.

= Tr 26, GS 2.12. For the same notation for n = 1, see MU 2.2.118, SGT pp. 15-17, GS 1.46, GL 30 (esp. lines 12-13 on p. 29).

Probably Govinda omitted the denominators when they were one and the same (say p):

$$\begin{array}{l} a_0 \\ a_1 \\ a_2 \\ \cdot \\ \cdot \\ \cdot \\ a_n \end{array}$$

This is a place-value notation for fractional parts with the base p. Cf. § 2.9 below.

2.3 Cancellation of the Greatest Common Factor

Reduction of two numbers to mutually prime ones.

tad uktaṃ govindasvāminā—

chedāṃśā <ṃ>ś cānyau vā chindyād anyonyabhaktaśeṣeṇa /

tatrāptau tāv eva dṛḍhāv idam apavartanaṃ karma //iti /[36]

"It has been said by Govindasvāmin—

"One should divide denominators *(cheda)* and numerators (*aṃśa*), or any other pair <of numbers>, by the <last> remainder <obtained when they are> mutually divided. The two obtained there are firm (*dṛḍha*) (i.e. mutually prime). This is a computation of reduction (*apavartana*)."

See § 1.1 above for Govinda's similar verse citied by Śaṅkara in the section for division (SL p. 20, lines 11-12). Śaṅkara states that Bhāskara II's rule (L 243) for the reduction (*apavartana*) of a and b in $y = (ax + c) / b$ to mutually prime numbers is applicable to any pair of numbers and cites this verse in support of his argument.

2.4 Addition and Subtraction

Śaṅkara cites a rule for addition and subtraction of fractions from Govinda's *Gaṇitamukha* (Introduction to Mathematics). This is the only place

[36] SL p. 61, lines 7-8.

where this work is referred to by name.

govindasvāminā́py uktaṃ gaṇitamukhe—

bhinnacchedāś chedair anyonyaṃ tāḍitāḥ samacchedāḥ /

aṃśāḥ sadṛśacchedāḥ saṃyogaviyogayogyās te //iti /[37]

"It has been told by Govindasvāmin, too, in <his> *Gaṇitamukha*—

"The <numerators> having different denominators, multiplied mutually by <other> denominators, have equal denominators; the different denominators, multiplied mutually by <other> denominators, are equal denominators.[38] Those numerators having equal denominators are to be employed for addition and subtraction."

That is to say, in modern notation.

$$\sum_{i=1}^{n}\frac{b_i}{a_i}=\sum_{i=1}^{n}\frac{b_i\prod_{j\neq i}a_j}{\prod_{j=1}^{n}a_j}=\frac{\sum_{i=1}^{n}\left(b_i\prod_{j\neq i}a_j\right)}{\prod_{i=1}^{n}a_i}.$$

In another place (SL p. 60, line 26), Śaṅkara cites the first line of this verse with his introduction, "That the dual number is not intended here (in addition and subtraction of fractions) has been shown by Govindasvāmin, too."[39] Bhāskara II's rule (L 30) for the same purpose is prescribed for two fractions[40] but Śaṅkara thinks that "the dual expression, 'of two <fractional> quantities' (*rāśyor*), in this case implies the plural also"[41] and cites the above rule of Govinda in support of this interpretation.

It is not known how Govinda expressed negative terms. In the *Bakhshālī Manuscript* and in the old anonymous commentary on Śrīdhara's *Pāṭīgaṇita,*

[37] SL p. 62, lines 9-10.

[38] I interpret the first line of the verse as having these double meanings since otherwise the rule is incomplete. The first meaning is obtained by regarding the first and the last compounds as *bahuvrīhi* and the second by regarding them as *karmadhāraya.*

[39] *atra dvivacanasyāvivakṣitatvaṃ govindasvāmināpi pradarśitam /*

[40] *anyonyahārābhihatau harāṃśau rāśyoḥ samacchedavidhānam evam / mitho harābhayām apavartitābhyāṃ yadvā harāṃśau sudhiyātra guṇyau //* L 30 /

[41] *atra rāśyor iti dvivacanaṃ bahūnām upalakṣaṇam /* (SL p. 60, line 16)

both of which were written in north-western India, a symbol like the modern symbol for addition,+, is placed next (right) to the number to be affected while in many other mathematical works a dot (•) or a small circle (o), both called *bindu* (lit. a dot), is used for negative numbers: it is placed on the right shoulder of the number in Bhāskara I's commentary on the *Āryabhaṭīya,* toward the left of the number in the Siṃhatilaka Sūri's commentary on Śrīpati's *Gaṇitatilaka,* and above the number in the works of other mathematicians including Bhāskara II. who was the most influential mathematician of medieval India.

For addition and subtraction of fractions, see AB 2.27, BSS 12.2, PG 32+ 36-37, Tr 19(= PG 32)+ 23, GP 15, GSS 3.55-56 MS 15.13-14, GT 32 (p. 15, lines 20-21)-33 (p. 18, lines 3-4)+ 46 (p.30, line 16)-47 (p. 31, lines 25-26), SS 13.8 + 12, L 30 + 37, GK 1.26 + 28 (p. 11, lines 6-7), GS 1.47-48 + 50, 2.1, GM 33 + 41.

2.5 Partial Addition and Subtraction

govindasvāmināpy uktam—

chedena jaghanyena cchedaṃ hatvānyam[42] *api hanyāt* /

sahitena ca rahitena svāṃśena svāṃśayutihānyoḥ // *iti* /[43]

"It has been said by Govindasvāmin, too—

"Having multiplied the denominator by the lower denominator, one should also multiply the other <member of the fraction> (that is, the upper fractional part) by <the lower denominator> increased or decreased by its own fractional part, in the case of addition and subtraction of its own fractional part."

This is to say,

$$\begin{bmatrix} b_1 \\ a_1 \\ \pm b_2 \\ a_2 \end{bmatrix} = \begin{bmatrix} b_1(a_2 \pm b_2) \\ a_1 a_2 \end{bmatrix} .$$

This is a rule for the so-called "partial addition" (*bhāga-anubandha)* to, and "partial subtraction" (*bhāga-apavāha*) from, a fraction, that is,

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42 *hatvāgryaṃ* SI /AB.

43 SL p. 70, lines 22-23.

$$\frac{b_1}{a_1} \pm \frac{b_1}{a_1} \cdot \frac{b_2}{a_2} = \frac{b_1(a_2 \pm b_2)}{a_1 a_2},$$

although these technical terms do not occur in Govinda's verses quoted. The word, *jaghanya* (lower), indicates that the two fractions are arranged vertically. See § 2.2 above for a similar arrangement of fractions in the case of an integer with fractional parts.

For partial addition and subtraction, see BM 25-26 + C4, BSS 12.9, PG 39-40, Tr 24 (= PG 39), GP 18, GSS 3.113 + 126, MS 15.11-12, GT 48 (p. 34, lines 15-16) + 51 (p. 37, lines 2-5), L 34, GK 1.27, GS 2.6 + 8, GM 37-38.

2.6 Multiplication

ata evoktaṃ govindasvāminā—

saty aṃśe guṇakāre tacchedo bhājakas tathā guṇye /

ubhayor apy aṃśakayoś chedahato bhājakaś chedaḥ[44] *// iti /*[45]

"Therefore, indeed, it has been said by Govindasvāmin—

"When the multiplier <in question> is a fractional part, its denominator is a divisor, <by which the product is divided>, and the same is also the case with the multiplicand. When both are fractional parts, the denominator multiplied by the <other> denominator is a divisor, <by which the product is divided>."

That is to say,

$$c \times \frac{b}{a} = (c \times b) \div a, \quad \frac{b}{a} \times c = (b \times c) \div a,$$

$$\frac{b_1}{a_1} \times \frac{b_2}{a_2} = (b_1 \times b_2) \div (a_1 \times a_2).$$

For multiplications of fractions, see BM Q3, BSS 12.3+8, PG 33+38 = Tr 20+23(cd) + 25 (ab), GP 16, GSS 3.2+99, MS 15.13+15, GT 35 (p. 19, line 26), SS 13.9+11, MU 2.2.120, L 32+39, GK 1.26+28, GS 1.53, 2.2, PV X14, GM 35+43.

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[44] *bhājakacchedaḥ* SL.

[45] SL p. 79, lines 2-3. The first line of this verse is cited also in SL p. 181, line 11.

2.7 Division

ata evoktaṃ govindasvāminā—

bhājye cāṃśe guṇito bhājyacchedena bhāgahāraḥ syāt /

aṃśe tu bhāgahāre bhājyaguṇo bhājakacchedaḥ // iti /[46]

"Therefore, indeed, it has been said by Govindasvāmin—

"When the dividend <in question> is a fractional part, the divisor will be multiplied by the denominator of the dividend. When the divisor is a fractional part, on the other hand, the denominator of the divisor is multiplied by the dividend."

That is to say,

$$\frac{b}{a} \div c = b \div (c \times a),\ c \div \frac{b}{a} = (a \times c) \div b.$$

For division of fractions, see BM Q4, BSS 12.4+60, PG 33+38 = Tr 20+23 (cd)+25 (ab), GP 16, GSS 3.8+99, MS 15.15+19, GT 36 (p. 21, lines 3-6)= SS 13.10, MU 2.2.121-123, L 41, GK 1.29 (p. 12, lines 11-12), GS 1.55, 2.3, PV X14, GM 43.

2.8 Square and Square-Root

govindasvāminā ca—

aṃśakṛtiṃ hṛtvāptaṃ kṛtyā chedasya bhinnavargaḥ[47] *syāt /*

aṃśasya mūlarāśeś chedapadenāpyate mūlam // iti /[48]

"And by Govindasvāmin—

"When one has divided the square of the fractional part <in question> by the square of <its> denominator, the quotient will be the square of the fraction. The root <of a fraction> is obtained from the root-quantity of the fractional part <divided> by the root of <its> denominator."

$$\left(\frac{b}{a}\right)^2 = b^2 \div a^2,\ \sqrt{\frac{b}{a}} = \sqrt{b} \div \sqrt{a}.$$

---

[46] SL p. 81, lines 21-22. The second line of this verse is cited also in SL p. 104, line 23 and p. 181, line 8.

[47] *rūpavargaḥ* SL.

[48] SL p. 86, lines 6-7.

For the squares and square-roots of fractions, see BSS 12.5, PG 34 =Tr 21, GP 17, GSS 3.13, MS 15.16, GT 38+40 (p. 23, line 25), SS 13.9, L 43, GK 1.29, GS 1.57+59, GM 44.

2.9 Square of an Integer with Fractional Parts

rūpakṛtau rūpadviguṇahatāṃśāc chedalabdham utkṣipya /
aṃśakṛteś chedāptaṃ pūrvāṃśe vā dhanaṃ[49] *vargaḥ //*[50]
iti bruvatā govindasvāminâpīdam eva sphuṭīkṛtam //

"The same has been made clear by Govindasvāmin, too, who says:

"When one has added the quotient of <the division of> the fractional part multiplied by twice the integer by the denominator to the square of the integer, the quotient of <the division of> the square of the fractional part by the denominator being added to the previous fractional part if possible, the square <of a number with fractional parts is obtained>."

That is to say (see §2.2 for the expression of fractional parts),

$$\begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} a_0^2 \\ 2a_0a_1 \\ a_1^2 \end{bmatrix}^2 = \begin{bmatrix} a_0^2 + (2a_0a_1 + a_1^2 @ p) @ p \\ (2a_0a_1 + a_1^2 @ p) \# p \\ a_1^2 \# p \end{bmatrix}$$

where the symbols, @ and #, in the expressions $a@b$ and $a\#b$ mean the quotient and the remainder in integers, respectively, of the division of a by b.

The word "same" in Śaṅkara's comment on the above quotation refers to the method he has just employed for the computation of the square of 3437;44,48, whose fractional parts are expressed in the sexagesimal notation.[51]

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49 *ghanaṃ* SL.

50 SL p. 88, lines 20-21.

51 The notational places are arranged vertically in the manuscripts used for Sarma's edition of the SL. See fn. 6 on p. 88 of his edition. The quantity, 3437 *kalās* 44 *vikalās* and 48 *tatparās*, is the radius of the reference circle of Mādhava's sine table (see NAB 2.12, p. 55). It is obtained by $r = C/2\pi$, where $C = 360 \times 60 = 21600$ *kalās*.

When $a_0 = 3437$, $a_1 = 44$, $a_2 = 48$, and $p = 60$,

$$\begin{bmatrix}3437\\44\\48\end{bmatrix}^2 = \begin{bmatrix}3437^2\\2\times3437\times44\\44^2+2\times3437\times48\\2\times44\times48\\48^2\end{bmatrix} = \begin{bmatrix}11812969\\302456\\331888\\371712\\2304\end{bmatrix} = \begin{bmatrix}11818102\\8\\39\\2\\24\end{bmatrix}$$

Śaṅkara gives this example for another rule equivalent to the above.

rūpakṛtim upari kuryād aṃśakṛtim adho dviguṇaghātam /

madhye tayor adhaḥsthāc chedāptaṃ kṣepyam upari muhur evam //[52]

"One should produce the square of the integer part above, the square of the fractional part below, and twice the product <of the two parts> in between the two. The quotient from <the division of> the lower by the denominator should be added to the upper. This <last procedure> is repeated many times."

Having explained Govinda's verse briefly, Śaṅkara also cites Lalla's verse for the same purpose, which is designed specifically for astronomy where unity is taken to be the *kalā* and the denominator (p), or the base, of the noational places of fractional parts 60:

lallenāpy ayam evopāyaḥ śiṣyadhīvṛddhidākhye mahātantre darśitaḥ / tathā ca tadvākyaḥ—

vikalākṛtitaḥ khaṣaḍḍhṛtaṃ[53] *svakalāghne*[54] *vikale dvisaṅguṇe /*

Mādhava's value of π, 2827433388233/(9 x10"), produces r = 3437; 44, 48, 22, 30,.... The same radius is obtained also by Śaṅkara's value of π (the value quoted by Śaṅkara, to be precise), 104348/33215 , which would produce r = 3437 ; 44, 48, 22, 24,.... The same, however, cannot be obtained by π = 355/113, which would produce r = 3437; 44, 47, 19,.... For Indian values of π, see Hayashi, et al., 1997, 301-398.

[52] SL p. 87, lines 21-22. The source of this verse is not known.

[53] *vikalasya kṛtiṃ khaṣaḍhṛtāṃ* in D, P, C. The abbreviations, D, P, and C, denote the texts of Lalla's *Śiṣyadhīvṛddhida* edited, in order, by Dvivedin (1886), Pandey (1981), and Chatterjee (1981).

[54] *svakalaghne* in D, P ,C.

viniyojya haren nabhorasaiḥ phalayuk pūrvakṛtiḥ kṛtir[55] *bhavet // iti /*[56]

"The same method has been taught by Lalla in <his> great treatise called *Śiṣyadhīvṛddhida.* Thus his statement is—

"Having added the quotient of the division of the square of *vikalā* by [sky, six] (60) to *vikala* (= *vikalā)* multiplied by its own *kalā* and two, one should divide <the sum> by [sky, tastes] (60). The square of the former (= *kalā*) is increased by the quotient, and there will be the square <of the number with sexagesimal fraction>."

For the same topic, see also BSS 12.62.

2.10 Square-Root of an Integer with Fractional Parts

tad uktaṃ govindasvāminā—

padaśeṣāc chedahatāt padadviguṇalabdhavargam atha śodhyam /

chedenāptaṃ śeṣād āpto 'ṃśaḥ sthūla itaro vā // iti /[57]

"It has been said by Govindasvāmin—

"The square of the quotient <of division> of the remainder of <subtraction of the square of> the square-root, multiplied by the denominator, by twice the square-root, divided by the denominator, should be subtracted from the remainder <of the division>. A fractional part <of the square-root> is obtained <in this way>. Or, in case it is rough, another <fractional part should be obtained in the same way>."

That is to say,

$$\left(a^2+\frac{2ab}{p}+\frac{b^2}{p^2}\right)-a^2=\frac{2ab}{p}+\frac{b^2}{p^2}.$$

$$\left(\frac{2ab}{p}+\frac{b^2}{p^2}\right)\times p\div 2a=b+\frac{b^2/p}{2a},\quad \frac{b^2}{p}-\frac{b^2}{p}=0\ .$$

Hence

$$\sqrt{a^2+\frac{2ab}{p}+\frac{b^2}{p^2}}=a+\frac{b}{p}.$$

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55 *phalayug rūpakṛtiḥ kṛtir* in SL, *phalayukyākṛtir* in P.

56 SL p. 89, lines 3-4 = SDV 4.51.

57 SL p. 90, lines 6-7.

This rule gives only the principle of the extraction of the square-root from an integer with fractional parts. In actual calculations, several supplementary steps are required. I provide here two examples for illustrating this rule. The calculations within <...> are not stated in the rule. The parts of the root obtained are underlined.

Ex. 1. To calculate the square-root of $\begin{bmatrix} 27 \\ 33 \\ 45 \end{bmatrix}$

Solution. $27 = \underline{5}^2 + 2$. 2×60 <+ 33> $= 153 = (2 \times 5) \times \underline{15} + 3$. $15^2 = 225 = 60 \times 3 + 45$. $3 - 3 = 0$ and <45 - 45 = 0>. Hence the answer is $\begin{bmatrix} 5 \\ 15 \end{bmatrix}$ Here, the integer part, 5, of the root must have been obtained by means of a mnemonic table of square-roots.[58]

Ex. 2. To calculate the square-root of $\begin{bmatrix} 11818102 \\ 8 \\ 39 \\ 2 \\ 24 \end{bmatrix}$

Solution. $11818102 = \underline{3437}^2 + 5133$. 5133×60 <+ 8> $= 307988 = (2 \times 3437) \times \underline{44} + 5532$. $44^2 = 1936 = 60 \times 32 + 16$. $5532 - 32 = 5500$ and $(39 - 16 = 23)$. At this stage of calculation, the root, $\begin{bmatrix} 3437 \\ 44 \end{bmatrix}$, is obtained but it is still rough because the original column of digits retain some of them: $\begin{bmatrix} 5500 \\ 23 \\ 2 \\ 24 \end{bmatrix}$ The remaining column, therefore is rewritten as: $\begin{bmatrix} 330023 \\ 2 \\ 24 \end{bmatrix}$,

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[58] For an example of such a table, see Sarma 1997.

where <330023 = 5500 × 60 + 23>, and the rule is applied again with the current "root", $\begin{bmatrix} 3437 \\ 44 \end{bmatrix}$. That is, 330023 × 60 <+ 2> = 19801382 = (2 × (3437 × 60 + 44)) × <u>48</u> + 38. 48^2 = 2304 = 60 × 38 + 24. 38 – 38 = 0 and <24 – 24 = 0>. Hence the answer is $\begin{bmatrix} 3437 \\ 44 \\ 48 \end{bmatrix}$. The integer part, 3437, of the root must have been obtained by means of the rule for the extraction of the square-root from a number expressed in a place-value notation (see § 1.3 above). The fractional parts, too, can be obtained by the same rule, and it is easier than the above.

3. Three-Quantity Operation (*Trairāśika*)

3.1 Definition

eteṣāṃ trayāṇāṃ lakṣṇam apy uktṇaṃ govindasvāminā—

yata idam āptam itīha vyapadeśas tat pramāṇam āptaṃ yat /

phalam icchā yad abhīṣṭaṃ yaj jijñāsyaṃ phalaṃ tasyāḥ // iti /[59]

"A definition, too, of these three <terms of a three-quantity operation> has been told by Govindasvāmin—

"When there is the information that 'This has been obtained from that', then it (the latter) is a *pramāṇa* (standard) and what has been obtained is a *phala* (fruit). What has been optionally desired is an *icchā* (requirement). What is to be inquired is a *phala* (fruit) of it (*icchā*)."

In his commentary on the *Mahābhāskarīya*, Govinda gives a more explanatory definition of the three-quantity operation.

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[59] SL p. 182, lines 26-27. The same verse is cited on p. 66, lines 8-9, too, with the following introduction: *pramāṇaphalecchārāśīnāṃ lakṣaṇam apy uktam /* (A definition of three quantities, <called> *pramāṇa, phala,* and *icchā,* too, has been told.)

katham idaṃ trairāśikaṃ nāma / idam iha trairāśikam /trayo rāśayaḥ samāhṛtāḥ kāraṇaṃ yasya sa rāśiḥ kārye kāraṇopacārāt trirāśir bhavati / sa prayojanaṃ[60] *yasya tad gaṇitaṃ trairāśikam / tatra pramāṇaṃ phalam icchā ceti trayo rāśayaḥ / teṣu tat pramāṇaṃ nāma yata idaṃ labdham iti vyapadiśati / labdhaṃ tu phalam / yat punar anena kiyal labhyata itīdam abhidhīyate tadicchā / yac ca punar jijñāsyaṃ tad icchāphalaṃ nāma / tatrecchāhataṃ phalaṃ pramāṇena vibhajet tadecchāphalāvāptiḥ /*[61].

"Why is this called *trairāśika?* The following is here the *trairāśika.* A quanity, the cause (*kāraṇa*) of which is three quantities collectively, is <called> *trirāśi* due to a figurative application of *kāraṇa* ("a cause") to *kārya* ("an effect"). A calculation whose object is that[62] <quantity called *trirāśi*> is *trairāśika.* There, the three quantities are a *pramāṇa* (standard, *a*), *a phala* (fruit, *b*) and an *icchā* (requirement, *x*). Among them, a *pramāṇa (a)* is that from which this (*b*) is said to have been obtained, whereas what has been obtained (*b*) is a *phala.* On the other hand, when <it is asked> how much of that (*y*) is obtained by means of that (*x*), this (*x*) is called an *icchā* of that (*y*). What is, on the other hand, to be sought is a *phala* for the *icchā.* There, one should divide the *phala* multiplied by the *icchā* by the *pramāṇa.* Then, one obtains the *phala* for the *icchā.*"

This prose passage, without the last sentence which prescibes the computational procedure, is cited by Śaṅkara,[63] and, with the last sentence, by Nīlakaṇṭha.[64] For the computational rule, see § 3.3 below.

The three given terms are usually arranged in a horizontal row, sometimes within three open cells or boxes, in the order of the *pramāṇa, phala,* and *icchā*:

a	*b*	*x*

[60] *tat samprayojanaṃ* (for *sa prayojanaṃ*) SL, *sa prayojanaṃ* SL/C.
[61] GMB 1.7, p. 8, lines 5-9.
[62] According to the reading of GMB and SL/C.
[63] SL, p. 178, lines 10-14.
[64] NAB 2.6, p. 32, lines 20-26.

Sometimes, however, they are arranged vertically (cf. Hayashi 1995, 413 and 2000, 185): $\begin{array}{|c|} a \\ b \\ x \\ \hline \end{array}$ We will express the relationship as :

$a : b = x : y.$

3.2 Three-Quantity Operation as a Kind of Inference

Govindasvāmin compares a *trairāśika* to an *anumāna* or inference of Indian logicians.[65]

ata eva trairāśikasyānumānarūpatāṃ pratijānatā govindasvāminoktam—

icchāpramāṇarāśyoḥ samānajātyor niyantṛdharmsya[66] /

gamako niyamyadharmo niyamena bhaved yatas tatas tena //

trairśikam anumānaṃ yata iha gamikā pramāṇajā saṅkhyā /

gamyā phalajā ca tayā pramāṇasaṅkhyā hi niyateyam[67]// *iti* /[68]

"Therefore, indeed, it has been said by Govindasvāmin who declares that a three-quantity operation has the form of inference—

"Due to a restricting rule (*niyama*), two quantities, *icchā* and *pramāṇa*, which are of the same kind, will have a quality to be restricted (*niyamya-dharma*), which is an indicator (*gamaka*) of a restricting quality (*niyantṛ-dharma*) <that exists in the *phalas*>, because of which, therefore, // a *trairāśika* is an inference (*anumāna*).

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65 In the stanzas cited here, Govindasvāmin attaches much importance to the concept of *niyama* (restricting rule) in *trairāśika* as well as in *anumāna*. This fact as well as his terminology such as *gamaka* and *gamya* presumably indicate that he was influenced by the Mīmāṃsā logicians. See, for example, the chapter called *Vyāptivāda* in Pārthasārathimiśra 's *Nyāyaratnamālā* (NRM). Nīlakaṇṭha, too, compares a *trairāśika* to an *anumāna* and cites stanza 3 of the *Vyāptivāda*. See NAB 2.12, p. 54, lines 13-14.

66 *niyatadharmasya* SL/C.

67 *niyamayate* SL/C.

68 SL p. 179, lines 18-21.

Table 5 : Comparison of *trairāśika* and *anumāna* acc. to Govindasvāmin.

anumāna	a kitchen		the mountain
	smoke^1 : fire^1	⇒	smoke^2 : fire^2
functions	sapakṣa dṛṣṭānta (Śaṅkara)		pakṣa dārṣṭāntika (Śaṅkara)
trairāśika	pramāṇa : p-phala	=	icchā : i-phala
	a palas : *niṣkas*	=	*x palas* : *y niṣkas*
	saffron I		saffron II

Table 6: Relations of the two general qualities involved in *anumāna.*

anumāna	smoke-ness	fire-ness
relations	sādhana-dharma gamaka → niyama niyamya-dharma ←	sādhya-dharma gamya niyantṛ-dharma
tairāśika	*a palas* of saffron	*b niṣkas*

"For, the indicator (*gamikā*) in this is the number produced as a *pramāṇa,* to be indicated (*gamya*) is the <number> produced as a *phala,* and this number for *pramāṇa* is indeed restricted by (i.e., concurs with) that <number produced as *phala*>."

tad apy uktaṃ govindasvāminā—

icchā svaphalaviśiṣṭā pakṣasapakṣau pramāṇarāśiś ca /

sādhyāṃśena samāno yo 'rthaḥ sa syāt sapakṣo hi //iti /[69]

"That, too, has been told by Govindasvāmin—

"The *icchā* characterized by its own *phala* and the quantity of *pramāṇa*

69 SL p. 180, lines 3-4.

<characterized by its own *phala*> are *pakṣa* (the subject) and *sapakṣa* (a similar instance), <respectively>. An object which is equal to part of what is to established (*sādhya*) shall be indeed a *sapakṣa.*"

A typical *anumāna* according to Indian logicians is as follows.

1. That mountain (*parvata*) has fire (*vahni*).
2. Because of its having smoke (*dhūma*).
3. That which has smoke has fire, like a kitchen (*mahānasa*).

The first half of the third statement, "That which has smoke has fire," is a "restricting rule" (*niyama*) in this case, which is established by means of "repeated observations" (*bhūyodarśana*) according to Śaṅkara. The correspondence between the four elements of an *anumāna* and those of a *trairāśika*, according to Govindasvāmin's three verses and Śaṅkara's comments on them, is shown in Tables 5 and 6, where I have supplied an example of purchase of saffron (*kuṅkuma*) as a typical case of the three-quantity operation (see L 74 for example). The *niyama* in that case would be its market price, that is, "*a palas* of of saffron are sold for *b niṣkas* and the price is proportional to the weight."

Note that the *pramāṇa* and *icchā* characterized by their respective *phalas* correspond to the *pramāṇa*-side and the *icchā*-side, respectively, of a *trairāśika,* as in the case of Śaṅkara's statement, "of the *pramāṇa* and *icchā,* which are *dṛṣṭānta* (an illustrative example) and *dārṣṭāntika* (that which is illustrated), <respectively>" (p. 179, lines 6-7). Cf. also the following statement of Śaṅkara:

atas tayoḥ samānajātīyatvaprayojako yo dharmaviśeṣaḥ
sādhyasāmānyavyāpta sādhanasāmānyarūpaḥ sa ubhayos sādhāraṇaḥ /
yathā vahnisāmanyavyāptaṃ dhūmasāmānyaṃ pakṣasapakṣayoḥ
parvatamahānasayoḥ sādhāraṇam /[70]

"Therefore, a particular quality, which causes the two (*pramāṇa* and *icchā*) to be of the same kind, and which has the form of 'a universal qualtity for establishing' pervaded by 'a universal quality to be established,' is

[70] SL p. 179, lines 10-12.

common to both (*pramāṇa* and *icchā*). It is just as the universal quality of smoke (*dhūma*) pervaded by the universal quality of fire (*vahni*) is common to the *parvata* and the *mahānasa*, which are *pakṣa* and *sapakṣa*, <respectively>."

The latter half of the third verse indicates that a *sapakṣa* (saffron I) is included in the extension of what possesses the *sādhya* ("price-ness").

3.3 Computational Rule

govindasvāmī ca—

trairāśikaphalarāśiṃ hatam icchārāśinā pramāṇena /

hṛtvecchāphalam āptaṃ syāt tat trairāśikagaṇitam // iti /[71]

"And Govindasvāmin <has said> —

"When one has divided the quantity of *phala* of a three-quantity operation multiplied by the quantity of *icchā* by the *pramāṇa*, the quotient will be the *phala* for the *icchā*. This is computation of a three-quantity operation."

That is to say,

$y = (b \times x) \div a.$

Govinda's verse closely resembles Āryabhṭa's:

trairāśikaphalarāśiṃ tam athecchārāśinā hataṃ kṛtvā /

labdhaṃ pramāṇabhajitaṃ tasmād icchāphalam idaṃ syāt //[72]

"When one has multiplied the quantity of *phala* of a three-quantity operation by the quantity of *icchā*, what has been obtained is divided by the *pramāṇa*. From it there will be this *phala* for the *icchā*."

For a comparison, I quote here two verses for the same purpose, one from Brahmagupta's BSS and the other from Śrīdhara's PG. Brahmagupta (born 598) and Śrīdhara (fl. 8th century) flourished between Āryabhaṭa (born 476) and Govinda (fl. 850). Their rules are identical with the above

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71 SL p. 182, lines 23-24. The same verse is cited on p. 66, lines 5-6, too, with the introduction, *yad uktam /* (since it has been said).

72 AB 2.26.

but their expressions are different from those of Āryabhaṭa and Govinda.

trairāśike pramāṇaṃ phalam icchādyantayoḥ sadṛśarāśī /

icchā phalena guṇitā pramāṇabhaktā phalam bhavati //[73]

"In a three-quantity operation, there are a *pramāṇa*, a *phala*, and an *icchā*, and in the first and the last <places> are like quantities. The *icchā*, multiplied by the *phala* and divided by the *pramāṇa*, becomes the *phala* <for the *icchā*>."

ādyantayos trirāśāv abhinnajātī pramāṇam icchā ca /

phalam anyajāti madhye tad antyaguṇam ādinā vibhajet //[74]

"In the first and the last <places> in a three-quantity operation are <located> *pramāṇa* and *icchā*, respectively, of the same category (*jāti*). A *phala* of another category is <located> in the middle. That (middle one) multiplied by the last, one should divide by the first."

See also BM C10 + N19 + Q11, GP 24, GSS 5.2, MS 15.24-25, GT 86 = SS 13.14, MU 2.2.113-115, L 73, GK 1.60, GS 1.63, PV X15, GM 88-89, CCM 32. Cf. the *Vedāṅgajyotiṣa*, verse 24 of the Ṛc recnsion and verse 42 of the Yajus recension (Sarma 1985).

3.4 Derivations of the Computational Rule

The six verses of Govinda cited in this section are most likely his metrical commentary on Āryabhaṭa's verse for the three-quantity operation (AB 2.26). See Derivations 2, 3, and 4 below. See §3.3 above for Āryabhaṭa's verse.

Derivation 1.

icchāphalasyānumitasya pramāṇasaṅkhyācchedatvaṃ tadanumānaṃ ca pradarśitaṃ govindasvāminā—

hatvā pramāṇaśuddhacchedenecchāṃ tato 'numīya phalam /

śakalīkṛtam ata āptaṃ chedenecchāphalaṃ bhavati //iti /[75]

"That the inferred *phala* for the *icchā* has the *pramāṇa* number as its

[73] BSS 12.10.

[74] PG 43 = Tr 29 (the latter reads *abhinnajāti* for *abhinnajātī* and *ādimena bhajet* for *ādinā vibhajet*).

[75] SL p. 180, lines 16-17.

divisor, as well as its inference, has been taught by Govindasvāmin—

"When one has multiplied the *icchā* by the *pramāṇa* which is a pure (integral) divisor and inferred a *phala* from it, what has been obtained from it when divided by the divisor becomes the *phala* for the *icchā*."

That is to say, in modern notation,

$$a : b = ax : bx = \frac{ax}{a} : \frac{bx}{a} = x : \frac{bx}{a} \rightarrow y = \frac{bx}{a}$$

Derivation 2.

ata evāha govindasvāmī—

phalatulyaṃ vā chedaṃ prakalpya bhāgapramāṇasya /

phalaguṇahārau tyaktvā tulyatvāt tat phalaṃ vāhuḥ // iti /[76]

"Therefore, indeed, Govindasvāmin has said—

"Or, having assumed the denominator of a fractional *pramāṇa* to be equal to the *phala*, and canceled multiplication and division of the *phala* because of the sameness, the revered teacher (Āryabhaṭa) has said <in his *Āryabhaṭīya* 2.26> that it is the *phala*."

In this verse, the two operations mentioned in Derivation 1 seem to be understood between the two explicitly stated operations, "having assumed" and "canceled." That is,

$$a : b = \frac{a}{b} : \frac{b}{b} \left(= \frac{a}{b} \times x : \frac{b}{b} \times x = \frac{\frac{a}{b} \times x}{\frac{a}{b}} : \frac{\frac{b}{b} \times x}{\frac{a}{b}} - \right) = x : \frac{bx}{a} \rightarrow y = \frac{bx}{a}.$$

To be "canceled" here is the division and multiplication by b in the last step of the above transformation:

$$\frac{\frac{b}{b} \times x}{\frac{a}{b}} = \left(\frac{b}{b} \times x\right) \times \frac{b}{a} = (((\underline{b \div b}) \times x) \underline{\times b}) \div a = bx \div a.$$

[76] SL p. 181, lines 15-16.

Derivation 3.

tad uktaṃ govindasvāminā—

etasyaivācāryāḥ pramāṇatulyaṃ prakalpya ca chedam /

icchātadguṇaharaṇe punar aviśeṣād vihāyāhuḥ // iti /[77]

"It has been said by Govindasvāmin —

"Having assumed the denominator of ths same thing (a fractional *pramāṇa)* to be equal to the <original> *pramāṇa,* and further canceled the multiplication and division of the *icchā* by it since there is no difference, the revered teacher (Āryabhaṭa) has said <in his *Āryabhaṭīya* 2.26 that it is the *phala*>."

Just as in Derivation 2,

$$a : b = \frac{a}{a} : \frac{b}{a} \left(= \frac{a}{a} \times x : \frac{b}{a} \times x = \frac{\frac{a}{a} \times x}{\frac{a}{a}} : \frac{\frac{b}{a} \times x}{\frac{a}{a}} \right) = x : \frac{bx}{a} \rightarrow y = \frac{bx}{a}.$$

Śaṅkara cites this verse before Derivation 2, but in Govinda's work the present verse must have followed Derivation 2 because the demonstrative pronoun, *etasyaiva,* in this verse refers to *bhāgapramāṇasya* in Derivation 2, and also because the object, *tat phalaṃ,* of the verb *āhuḥ,* is omitted in this verse, while it is explicitly stated in Derivation 2.

Derivation 4 and paraphrase of AB 2.26.

tad apy uktaṃ govindasvāminā —

icchāpramāṇabhāgair yāvadbhiḥ saṃyutaṃ bhvati dṛṣṭam /

yuktaṃ tatphalabhāgais tāvadbhis tatphalaṃ bhavati //

iti conniyācāryāḥ[78] *kurvantīcchāguṇasya phalarāśeḥ*[79] */*

icchāphalāvagatyai haraṇaṃ tena pramāṇena //

athavā phalaguṇitecchāpramāṇabhāgāh phalaṃ bhaved iṣṭam /

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[77] SL p. 181, lines 3-4.

[78] *vonnīyā-* SL/D.

[79] *kurvantīcchāphalasya guṇarāśeḥ* SL/A.

tat punar icchārāśau pramāṇakasamānajātīye / iti /[80]

"That, too, has been told by Govindasvāmin —

"When a *dṛṣṭa* ('seen', = *pramāṇa)* is joined with the quotients of division of the *icchā* by the *pramāṇa,* its *phala* is joined with the same number of the quantity[81] of its *phala.//* Having inferred in this way, the revered teacher makes division of the quantity of *phala* multiplied by the *icchā* by the *pramāṇa* in order to obtain the *phala* for the *icchā //* Or, the *icchā* multiplied by the *phala* and divided by the *pramāṇa* will be the desired *phala,* <since the exchange of the multiplier and the multiplicand does not cause any difference in the product.> And this is when the quantity of the *icchā* is of the same category as the *pramāṇa."*

The first verse presumably provides the fourth derivation:

$$a : b = a \times \frac{x}{a} : b \times \frac{x}{a} = x : \frac{bx}{a} \rightarrow y = \frac{bx}{a},$$

but this interpretation is not decisive because the meanings of the words, *yukta /saṃyuta* ("joined", usually used for addition in mathematics) and *dṛṣṭa,* are not certain. According to Śaṅkara, this verse means the transformation:[82]

$$a : b = 1 : \frac{b}{a} = 1 \times \frac{x}{\frac{a}{a}} : \frac{b}{a} \times \frac{x}{\frac{a}{a}} = x : \frac{b}{a} \times x = x : \frac{bx}{a}$$

but it is impossible to read the verse in this way.

The second verse paraphrases Āryabhaṭa's verse, which prescribes:

$$y = \frac{b \times x}{a}.$$

[80] SI. p. 181, line 23 - p. 182, line 3 and line 6. The last line is quoted by Śaṅkara separately with the introduction: *tad etatsarvam icchāpramāṇarāśyoḥ samānajātīyatve saty evopapadyata ity uktam* ("All this is justified only when the two quantities, *icchā* and *pramāṇa,* have the state of being of the same category. With this in mind, <the following half stanza> has been told."). Although this introduction does not contain the name of Govinda, there is no doubt that the line immediately follows the preceding half stanza.

[81] Read *tatphalamānais* instead of *tatphalabhāgais.*

[82] The demonstrative pronoun, "that", in his introduction points to this transformation.

The former half of the third verse gives an alternative rule by exchanging the multiplicand and the mutiplier:

$$y = \frac{x \times b}{a},$$

while the latter half provides the condition for the three quantities of a three-quantitry operation, that the first and the last quantities (a and x) are of the same category, a condition which was explicitly prescribed by Brahmagupta, Śrīdhara, and others but not by Āryabhaṭa. See §3.3 above.

3.5 Easy Method of Calculation

In his commentary on MB 1.23, Govindasvāmin prescribed a formula for expanding the fraction, b/a, in order to make the actual calculation of bx/a easier when a and b are large numbers. Since it is an important technique in the actual application of a three-quantity operation, I quote it here although Śaṅkara does not.

bhājakād[83] *guṇakāreṇa nihatād yena kenacit* /
bhājako guṇakārād vā bhājakenāpyate guṇaḥ //
matir bhavati sā saṃkhyā hartavyā[84] *hanyate yayā* //
matir anyatvam āpnoti phalataḥ khaṇḍanaṃ prati //
hīnāṃśe 'ṃśaḥ[85] *phale śeṣo 'dhikāṃśe tv adhiko bhavet* /
chedo hārahato hāro guṇahārau ca tau dṛḍhau //
tābhyām āptaṃ phale hāre pūrvalabdhād[86] *ṛṇaṃ dhanam* //
vyatyayād[87] *guṇakāre tu guṇyam ekam ihocyate* //[88]

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[83] *bhājakaṃ* in GMB/BC. The abbreviations, GMB/A, GMB/B, etc. denote the mss. used by Kuppanna Sastri for his edition of the GMB.

[84] *hartavyo* in PGMB and NAB; *kartavyo* in NAB/K. The abbreviations, NAB/K, NAB/Kh, etc., denote the mss. used by Sāmbaśiva Śāstrī for his editon of the NAB.

[85] *hīnāṃśakaiḥ* in GMB/A.

[86] *labdham* in GMB/A.

[87] *vyatyāsa* in GMB/BC.

[88] GMB 1.23, pp. 33-34.

"From the divisor multiplied by a certain <optional quantity>, by means of <division by> the multiplier, a <new> divisor is obtained; or, otherwise, from the multiplier multiplied by a certain <optional quantity>, by means of <division by> the divisor, a <new> multiplier is obtained.// The number by which that which is to be divided[89] is multiplied is "an intelligence number" (*mati*). The intelligence number obtains the state of being the other side of the result (quotient) with regard to the division (a divisor in the former case and a multiplier in the latter). // When the quotient lacks a fraction, the numerator <of the second term> will be the remainder <of the division>, but, when it contains an additional fraction, it will be the additional term. The denominator is the <first> divisor multiplied by the <second> divisor. Those two made firm (i.e. made relatively prime) are the <second pair of> multiplier and divisor.// What is obtained by means of the two <numbers> is <in order> subtracated from, or added to, the previous result when the quotient is a divisor, but <it is treated> inversely when it is a multiplier. The multiplicand in this case is said to be one and the same."

When m is any optional number and $\frac{am}{b} = q \pm \frac{r}{b} \; (0 \leq r < b)$,

$$\frac{b}{a} = \frac{m}{\frac{am}{b}} = \frac{m}{q} \mp \frac{r}{aq}.$$

Similarly, when $\frac{bm}{a} = q \pm \frac{r}{a} \; (0 \leq r < a)$,

$$\frac{b}{a} = \frac{\frac{bm}{a}}{m} = \frac{q}{m} \pm \frac{r}{am}.$$

Hence follows, in the former case,

$$y = \frac{bx}{a} = \frac{x \times m}{q} \mp \frac{x \times r}{aq},$$

and, in the latter case,

$$y = \frac{bx}{a} = \frac{x \times q}{m} \pm \frac{x \times r}{am}.$$

[89] Read *hartavyo* as in PGMB and NAB.

The x is called 'the multiplicand" in the last sentence of the cited passage. Note that if one chooses an m in such a way that r is small enough in comparison with aq in the first case and with am in the second, then one can ignore the second term in approximate calculations, that is,

$$y=\frac{bx}{a}\approx\frac{x\times m}{q},\quad\text{and}\quad y=\frac{bx}{a}\approx\frac{x\times q}{m}.$$

This seems to be the reason why the m is called "an intelligence number." In this way one can avoid a pair of multiplicantion and division by large numbers.

It is by means of this rule that Govindasvāmin explains the two pairs of "multiplier and divisor" or the two fractions, 29 / 36 and 43 / 72000, employed in the rule of MB 1.23. According to Āryabhaṭa, the number of omitted *tithis* (lunar days) and that of solar years in a *yuga* are 25082580 and 4320000, respectively. Let x be the number of solar years elapsed from the beginning of the current Kaliyuga. Then the number of the elapsed omitted *tithis* is:

$$y=\frac{25082580x}{4320000}=\frac{1254129x}{216000}=5x+\frac{174129x}{216000},\ \text{where}$$

$$\frac{174129}{216000}=\frac{29}{\frac{216000\times 29}{174129}}=\frac{29}{36-\frac{4644}{174129}}=\frac{29}{36}+\frac{4644}{216000\times 36}=\frac{29}{36}+\frac{43}{2000\times 36}.$$

It is therefore reasonable to say that this technique was known to Bhāskara I, if not to Āryabhaṭa I.

Commenting on this rule (PGMB 1.23, pp. 32-34), Parameśvara calls it "an easy method" *(laghu-tantra)*, and so also does Nīakaṇṭha, who cites, in his commentary on the *Āryabhaṭīya* (NAB 2.12, pp. 53 and 57), only the first two verses, which state: $\frac{bx}{a}=\frac{x\times m}{am/b}=\frac{x\times(bm/a)}{m}$.

3.6 Inverse Three-Quantity Operation

The inverse three-quantity operation is applied if a certain amount of an object measures b when measured by a measure (or unit) of size a, and measures y when measured by another measure (or unit) of size x, that is, $ab = xy$.

tad uktaṃ govindasvāminā —

bhinnapramāṇapramite 'nyena[90] *pramīyamāṇe 'rthe /*

guṇakṛt pramāṇam icchā hāras trairāśikaṃ vyastam // iti /[91]

"It has been said by Govindasvāmin —

"When an object, which has been measured by a measure of different standard, is being measured by another <measure>, the *pramāṇa* is a multiplier and the *icchā* is a divisor. <This is> the inverse three-quantity operation."

That is to say,

$$y = (a \times b) \div x.$$

I express the relationship underlying this computation as:

$$a :: b = x :: y.$$

Śaṅkara provides four examples for the inverse three-quantity operation. The first three, given in verse, are concerned with commercial problems: measurement of grain with different measuring cups,[92] that of gold with a unit (*māṣa*) having different standards,[93] and cutting of a blanket into pieces of different sizes.[94] The remaining one is concerned with astronomy: the relationship between the mean anomaly ($\overline{\alpha}$) and the distance (R) of a planet and its refined anomaly (α) and distance (ρ). Since $R \sin \overline{\alpha} = \rho \sin \alpha$, we have the relationship:

$$R :: R \sin \overline{\alpha} = \rho :: R \sin \alpha.$$

Śaṅkara cites a prose passage from Govinda's commentary on the MB[95]

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90 *-pramitānyena* SL/C.

91 SL p. 183, lines 8-9.

92 SL p. 190, lines 2-3. Cf. PG Ex. 34 = Tr Ex. 38. = Tr Ex. 40, the problem in which is numerically equivalent to the one cited by Śaṅkara.

93 The verse is almost the same as PG Ex. 35 = Tr Ex. 41.

94 The verse is almost the same as PG Ex. 37 = Tr Ex. 43.

95 SL p. 190, lines 19-21 = GMB 4.21, p. 197, lines 4-7.

and a verse from his *Govindakṛti,* both of which utilize the above relationship. The latter reads as follows:

govindakṛtāv[96] *api —*

trijyāhatakendrabhujāṃ karṇena hared bhujā bhaved āptā /

pratimaṇḍalasañjātā jyātaḥ kāṣṭhaṃ tataḥ kuryāt // iti /[97]

"In <his> *Govindakṛti* also —

"One should divide the sine of a mean anomaly <of a planet> multiplied by the sine of three <signs> by the 'ear' (refined distance of the planet). The quotient will be the sine produced on the 'counter circle' (eccentric circle). Then, from the sine <obtained>, one should produce the arc <corresponding to it by means of a since table>."

That is to say,

$$R \sin \alpha = \frac{R \times R \sin \overline{\alpha}}{\rho}, \quad R \sin \alpha \rightarrow \alpha \text{ (by a sine table)}.$$

For the inverse three-quantity operation, see BSS 12.11, PG 44(cd) = Tr 30, GSS 5.2, MS 15.25, L 73 + 77-78, GK 1.61, GS 1.81, PV X15, GM 96-97, CCM 32.

3.7 Double-Three-Quantity Operation

A double-three-quantity operation consists of two consecutive three-quantity operations. It is usually called "a five-quantity operation" (*pañcarāśika)* because the operation yields an answer from five quantities given. Nīlakaṇṭha cites Govinda's rule for that operation.

ata evaikasmin viṣaye 'nekatrairāśikasannipāte lāghavāyāha govindasvāmī —

guṇadvayasya saṃvargo bhāgahāradvayasya ca /

guṇako bhāgahāraś ca syātāṃ trairāśikadvaye // iti /[98]

"Therefore, indeed, for the sake of easiness <of calculations> when more than one three-quantity operations are combined in one object,

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96 *govindasvāmikṛtāv* SL/A.

97 SL p. 190, lines 22-23.

98 NAB, p. 14, lines 12-13.

Govindasvāmin has said—

"The product of the two multipliers and <that> of the two divisors will be a multiplier and a divisor, <respectively>, in the case of a double-three-quantity operation."

When two consecutive three-quantity operations are involved in one problem, that is,

$$a_1 : b = x_1 : y_1,$$

$$a_2 : y_1 = x_2 : y,$$

then

$$y_1 = b \times x_1 \div a_1,$$

$$y = y_1 \times x_2 \div a_2 = (b \times x_1 \div a_1) \times x_2 \div a_2 = b \times (x_1 \times x_2) \div (a_1 \times a_2).$$

In general, when n three-quantity operations are combined in one object, that is,

$$a_1 : b = x_1 : y_1,$$

$$a_2 : y_1 = x_2 : y_2,$$

$$a_3 : y_2 = x_3 : y_3,$$

$$\vdots \quad \vdots \quad \vdots$$

$$a_n : y_{n-1} = x_n : y,$$

then

$$y = b \times \frac{x_1 . x_2 \ldots x_n}{a_1 . a_2 \ldots . a_n}.$$

This is equivalent to the operation usually called "the $(2n + 1)$ – quantity operation" (*pañcarāśika, sapta-rāśika, nava-rāśika,* etc.), which has been prescribed as follows. (1) Arrange the $(2n + 1)$ quantities in two columns called the *pramāṇa*-side and the *icchā*-side. (2) Exchange the *phalas* of both sides (b and the unknown quantity). (3) Divide the product of the longer side by that of the shorter side. The quotient is the *phala*.

$$\begin{matrix} a_1 & x_1 \\ a_2 & x_2 \\ \vdots & \vdots \\ a_n & x_n \\ b & \end{matrix} \rightarrow \begin{matrix} a_1 & x_1 \\ a_2 & x_2 \\ \vdots & \vdots \\ a_n & x_n \\ & b \end{matrix} \rightarrow y = \frac{x_1 x_2 \cdots x_n b}{a_1 a_2 \cdots a_n}$$

For the $(2n + 1)$ – quantity operation, see BSS 12.11-12, PG 45 = Tr 31, GP 24, GSS 5.32, MS 15.26-27, GT 97 = SS 13.15, MU 2.2.115-116, L 82, GK 1.62, GS 1.72, GM 90, CCM 33+35.

Abbreviations of Titles

AB = Āryabhaṭa I's *Āryabhaṭīya*. [Kern 1973, Sāmbaśiva 1930, Shukla 1976]

BAB = Bhāskara I's *bhāṣya* on the AB. [Shukla 1976]

BG = Bhāskara II's *Bījagaṇita*. [Āpaṭe 1930]

BM = *Bakhshālī Manuscript*. [Hayashi 1995]

BSS = Brahmagupta's *Brāhmasphuṭasiddhānta*. [Dvivedī 1902, Sharma 1966]

CCM = Giridharabhaṭṭa's *Caturacintāmaṇi*. [Hayashi 2000]

GK = Nārāyaṇa's *Gaṇitakaumudī*. [Dvivedī 1936/42]

GL = Gaṇeśa I's *Buddhivilāsinī* on the L. [Āpaṭe 1937]

GM = Gaṇeśa II's *Gaṇitamañjarī*. [Hayashi n.d.]

GMB = Govindasvāmin's *bhāṣya* on the MB. [Kuppanna 1957]

GP = Śrīdhara's *Gaṇitapañcaviṃśī*. [Pingree 1979]

GS = Ṭhakkura Pherū's *Gaṇitasāra*. [Agaracanda and Nāhaṭā 1961]

GSS = Mahāvīra's *Gaṇitasārasaṃgraha*. [Raṅgācārya 1912, Jain 1963]

GT = Śrīpati's *Gaṇitatilaka*. [Kāpadīa 1937]

L = Bhāskara II's *Līlāvatī*. [Āpaṭe 1937, Sarma 1975]

MB = Bhāskara I's *Mahābhāskarīya*. [Kuppanna 1957, Shukla 1960]

MS = Āryabhaṭa II's *Mahāsiddhānata*. [Dvivedī 1995]

MU = Someśvara's *Mānasollāsa*. [Shrigondekar 1925/61]

NAB = Nīlakaṇṭha's *bhāṣya* on the AB. [Sāmbaśiva 1930]

NRM = Pārthasārathimiśra's *Nyāyaratnamālā*. [Gaṅgādhara 1900, Subrahmanya 1972]

PG = Śrīdhara's *Pāṭīgaṇita*. [Shukla 1959]

PGMB= Parameśvara's super commentary on the GMB. [Kuppanna 1957]

PV = *Pañcaviṃśatikā*. [Hayashi 1991]

SDV = Lalla's *Śiṣyadhīvṛddhidatantra*. [Chatterjee 1981, Dvivedī 1886, Pandey 1981].

SGT = Siṃhatilaka Sūri's commentary on the GT. [Kāpadīā 1937]

SL = Śaṅkara Vāriyar's part of the *Kriyākramakarī* on the L. [Sarma 1975]

SS = Śrīpati's *Siddhāntaśekhara*. [Miśra 1932/47]

Tr = Śrīdhara's *Triśatikā*. [Dvivedī 1899]

TS = Nīlakaṇṭha's *Tantraṃgraha*. [Sarma 1977]

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