A COMMENTARY OF TANTRASAṄGRAHA IN KERALABHĀṢĀ: KRIYĀKALĀPA

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There has been considerable confusion on whether the text of *Tantrasaṅgraha* itself includes the famous verses on the various infinite series for $\frac{\pi}{4}$ as also the series for sine and cosine functions [17]. Much of this confusion originates from Whish's paper itself, wherein verses outlining these series have been ascribed to *Tantrasaṅgraha*. Our study of the Whish manuscript of the Malayalam commentary (*Kriyākalāpa*) of *Tantrasaṅgraha* clearly establishes that these verses are all citations made in the commentary by the commentator Śaṅkara and are not part of *Tantrasaṅgraha*, a point which has been earlier emphasized by K. V. Sarma based on a careful study of several manuscripts of *Tantrasaṅgraha*.[1]

1 Introduction

It is well known that mathematicians and astronomers residing on the banks of the river Nila in the south Malabar region of Kerala around 14th century had advanced to the point of discovering the series expansion of $\frac{\pi}{4}$, (popularly known as Gregory series), and several rapidly converging approximations to that.[2] This was first brought to the notice of modern mathematicians by C. M. Whish[3]—a civil servant of the East India Company—during 1830s through his pioneering studies on four works. These are indeed four seminal works of Kerala school of astronomy, namely *Tantrasaṅgraha*, *Yuktibhāṣā*, *Karaṇapaddhati* and *Sadratnamāla*. Incidentally, it is quite interesting to note those works being referred to by Whish, as four *Śāstras* in his paper, where he also gives a detailed account

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[1]We plan to deal with this issue in greater detail in a future publication.

[2]See, for instance, Plofker [15] presenting an overview of Mathematics in India or [6] giving a detailed mathematical exposition of the celebrated text *Gaṇita-Yuktibhāṣā*.

[3]See [21]. His work somehow seems to have gone unnoticed for almost a century till C. T. Rajagopal and others took note of it in 1940s [16]. Today, of course, every historian of mathematics is aware of this "negleted chapter" in the history of mathematics.

of various series dealt with in them. Whish also notes towards the end of his article that:

> ...the copy of the work, which I have obtained with some difficulty, by frequent intercourse with this interesting society, bears in itself marks of antiquity, the commentary in the vulgar tongue being written in a language which is not now current in Malabar, and forms of many letters differing materially from those of the present day.[4]

The present paper reproduces a section of the commentary written in *Keralabhāṣā* (Malayalam language)—referred as "vulgar tongue" by Whish in the above citation. In particular, we shall present the commentary on the verses 3–6 of the second chapter of *Tantrasaṅgraha* where the commentator discusses the infinite series for $\frac{\pi}{4}$ and some of its rapidly convergent transformed versions. But before doing so, we present the details of the available manuscripts and also discuss about the authorship of this commentary as well as its title *Kriyākalāpa*.

1.1 Manuscript material

Our edition of the Malayalam commentary (*Kriyākalāpa*) of *Tantrasaṅgraha* is based on following three manuscripts which are being referred to as A, B and C.

A. Ms. 697 of the Oriental Research Institute and Manuscript Library, University of Kerala (KUOML) is a palm-leaf manuscript, inscribed in Malayalam script. It has 214 folios, containing about 8 lines per page. Both the sides of the folio have been inscribed. The dimension of the folio is approximately 24.5 cm $\times$ 3.5 cm. Though the parts of some folios have been worm eaten the manuscript is considerably in good shape and the letters are clearly visible (see Figures 1 and 2 displaying a few folios). There is a title page which mentions that the manuscript contains commentary of *Tantrasaṅgraha* in Malayalam language (*keralabhāṣāvyākhyopeta*). It also mentions that it has been inscribed in the Malayalam era 920, which corresponds to 1745 CE, and that it belongs to *Kudallur mana* (a family belonging to Kudallur).

B. This is a paper manuscript available at the Royal Asiatic Society (RAS), London.[5] While there are several catalogued manuscripts belonging to the

[4] See [21], p. 522.

[5] One of the authors of the paper (KR) could by chance lay his hands on this manuscript, while trying to trace Ms. 134 in *A catalogue of South Indian Manuscripts (especially those of Whish collection) belonging to Royal Asiatic Society of Great Britain and Ireland* (1902,

collection of Whish in RAS, there also seem to be a few uncatalogued ones. The paper manuscript used in the preparation of this paper (photo-images of a couple of its folios are also presented later in the paper) may be described as "*Tantra Saṅgrahaṃ*, RAS Whish papers [uncatalogued]". This manuscript contains 182 folios, each folio having about 24 lines. The text is quite readable. However, the manuscript is not complete and abruptly ends in the fifth chapter of *Tantrasaṅgraha*, with the noting at the end: "vide other book for the remainder." On the opposite page there is a signature of Whish with the year marked as 1820. There is a seal of "Madras Literary Society" on the margin of the first page of the manuscript.

C. This is a paper manuscript of *Kriyākalāpa* available among the K. V. Sarma collections, which is currently in possession of Sree Sarada Educational Society, Adyar, Chennai. This manuscript as per the notings of Sarma seems to be a paper transcript of the palm-leaf manuscript A, described above, on which some editoral work has been done

While manuscripts A and B do not carry any punctuation marks whatsoever, we found a few in manuscript C. In the edition that is being presented in section II, we have retained these punctuation marks as well as added a few to enhance the readability. It may also be mentioned here, that among the three manuscripts employed by us, by and large we found B (though incomplete) to be providing the most accurate version—free from scribal errors and other lapses.

1.2 Textual presentation

The edited version of the manuscript material presented in section II includes Sanskrit verses and Malayalam commentary. We have adopted the following editorial style in presenting the material.

1. The Sanskrit verses either from the text *Tantrasaṅgraha* or those that have been cited in the commentary are given using English alphabets with diacritical marks. In doing so, we have adopted the standard style of splitting the *sandhis* wherever possible. For example, we have typed *bhaved hāraḥ* instead of *bhaveddhāraḥ*.

2. To distinguish the verses of *Tantrasaṅgraha* from those that have been cited by Śaṅkara in his commentary, we have used bold fonts for the former while reserving the normal ones for the latter.

p. 190) compiled by M. Winternitz. The catalogued Ms. 134, which has been identified by K. V. Sarma as '*Tantrasaṅgraha* with the commentary *Laghuvivṛti*' [18] p. lviii, however, was not accessible at that time.

3. The commentary in the Malayalam language is given using the Malayalam script itself. The alternative readings found among the manuscripts are indicated in the footnotes.

4. While the Mss. A and B do not employ two different characters or notations to represent the short (*hrasva*) and the long (*dīrgha*) forms of the vowels '*e*' (എ) and '*o*' (ഒ), C. does. For instance, the word '*yoga*' (which means 'the sum') is represented in A and B as 'യൊഗ' whereas in C it is written as 'യോഗ'. Similarly, certain markers at the end of the word[6] that were found to be missing in Mss. A and B, were present in C. For example, the string to describe an operation 'with/by nine' is represented as 'ഒമ്പതു കൊണ്ട' in A and B, whereas it is written as 'ഒമ്പതു കൊണ്ടു(ണ്ട്)' in C. In presenting a critical edition of the text, we decided to follow the style adopted in Ms. C, which also by and large happens to be the style in vogue too.

5. In all the three manuscripts, the Sanskrit words like '*varga*' and '*caturtha*'—that include the character '*r*'—are written as '*vargga*' (വർഗ്ഗ) and '*caturttha*' (ചതുർത്ഥ), with the duplication of the consonant that follows it. Since, such a duplication is generally not indicated while writing the same Sanskrit words using *Devanāgarī* script, we, while presenting the Sanskrit verses in their transliterated forms, have not represented them in their duplicated forms. However, we have retained such forms in the Malayalam commentary.

1.3 Authorship of the commentary

The chapter-ending colophons in the Malayalam commentary presented in the paper reads as follows:

> *iṅṅine tantrasaṅgrahattile prathama(raṇṭām)addhyāyattile arttham collītāyi.*
>
> Thus has been explained the content of the first(second) chapter of *Tantrasaṅgraha*.

Unlike colophons in certain other texts, where one may find some biographical details of the author, here they are quite simple and straight forward. The author of this commentary seems to be reticent even to mention his name in

[6]These markers are generally employed to connote the meaning of 'having accomplished' a certain process or mathematical operation, which forms a part of a much larger computational procedure or, doing something 'with' something.

the colophon. However, from other internal evidences available, it can be safely concluded that the author of this Malayalam commentary is Śaṅkara Vāriyar (1500-60 CE).

Sarma by an extensive analysis of the expressions employed in the invocatory and introductory verses of the three commentaries—*Yuktidīpikā*, *Kriyākramakarī* and *Laghuvivṛti*—in his inroduction to the *Tantrasaṅgraha*[7] has brilliantly brought out the characteristic similarities suggesting the identity of the authorship of all the three works. Given that such an analysis has been made and it has been conclusively proved that the author of all the three commentaries is Śaṅkara, it has become all the more easier for us to conclude that the author of *Kriyākalāpa* must be Śaṅkara, since the invocatory verse of this work happens to be identical with that found in the *Laghuvivṛti* and runs as follows:

> *pratyūhavyūhavihatikārakaṃ paramaṃ mahaḥ* |
> *antaḥkaraṇaśuddhiṃ me vidadhātu sanātanam* ||
>
> May the eternal supreme effulgence, that destroys the arrays of obstacles, kindly purify my intellect.[8]

1.4 Title of the commentary: *Kriyākalāpa*

The term "*kriyākalāpa*" repeatedly occurs in the colophonic verses of Śaṅkara's famous Sanskrit commentary, *Laghuvivṛti*, of *Tantrasaṅgraha*. These verses composed in *Āryā* metre, with minor variations in different chapters, read as follows:

> *iti tantrasaṅgrahasya "kriyākalāpaṃ" krameṇa saṅgṛhya* |
> *racite tadvyākhyāne prathamo'dhyāyaḥ prapūrṇo'bhūt* ||
>
> Thus the commentary of the first chapter of *Tantrasaṅgraha*, wherein we have systematically and concisely presented its collocation of calculations (*kriyākalāpa*), has come to an end.

The quaintness of the expression (*kriyākalāpa*) appearing in the above verse, particularly in combination with the word *saṅgṛhya*, and that too in the colophonic verse, is likely to generate a doubt in the minds of the reader: "whether the word *kriyākalāpa* here refers to a (larger) commentary by name *Kriyākalāpa*,

[7] See [18], Introduction pp. liv–lvi.

[8] This *maṅgalācaraṇa*, though not explicitly mentioned, is addressed towards Lord Gaṇeśa, since he is considered to be the remover of the obstacles (*vighneśa*). Keeping in line with the tradition here Śaṅkara prays to Gaṇeśa to bless him with uninterrupted flow of thought that is essential for the completion of the work undertaken.

or does it merely refer to the collocation of calculations described in the text *Tantrasaṅgraha*". As the verse clearly lends itself to both kinds of interpretations, *a priori* there is no reason to reject or adopt either of them. In his introduction to *Kriyākramakarī*,[9] published in 1975, Sarma observed:

> He [Śaṅkara] names his larger commentary as *Kriyākalāpa* in the several chapter ending colophons of the *Laghuvivṛti*.

However, in his introduction to the edition of *Tantrasaṅgraha*[10] with *Yuktidīpikā* published in 1977, Sarma seems to have changed his view.

> ...the expression *tantrasaṅgraha-kriyākalāpaṃ* found in the said colophonic verse at the close of the different chapters of *Laghuvivṛti*, has only its literal meaning, viz., 'the collocation of computations of the *Tantrasaṅgraha*' and does not refer to any commentary on the work nor to any part of the work. The larger commentary implied in the introductory verse is to be identified with TS com. *Yuktidīpikā* referred to Śaṅkara himself in the *Laghu.* on TS. VII.4–6 (p. 322 below).

Subsequently, based on his study of the paper transcript C. of the KUML palm-leaf manuscript A., Sarma seems to have again revised his opinion. The paper transcript C. has been given the title *Kriyākalāpa* by Sarma (*kriyākalāpākhyayā keralabhāṣāvyākhyayā sameta*), perhaps as a first step towards preparing a final edited version of the work. This seems to be entirely reasonable as the Malayalam commentary (as we shall see in the specific example to be discussed below) is indeed much more extensive than *Laghuvivṛti*. So, Śaṅkara seems to be clearly referring to this Malayalam commentary "*Kriyākalāpa*" when he states "*iti tantrasaṅgrahasya "kriyākalāpaṃ" krameṇa saṅgṛhya*" in his *Laghuvivṛti* at the end of each chapter of *Tantrasaṅgraha*.

1.5 Organization of the paper

The rest of the paper is organized into three sections. Section II presents verses 3–6 of the II chapter (*sphuṭaprakaraṇam*) of *Tantrasaṅgraha*—dealing with the procedure for obtaining the chord from the arc of a circle (*jyānayanam*)—and the detailed Malayalam commentary that has been written by Śaṅkara on these verses. Apart from explaining the textual verses, here the commentator Śaṅkara makes a long excursus to discuss the relationship between the circumference and

[9] See [8], p. xix.
[10] See [18], pp. lviii–lix.

the diameter of the circle in great detail. This long excursus includes citations of verses which give Mādhava series and its several transformed versions, that have rapid convergence.

In section III, we provide an English translation of both the Sanskrit text and the Malayalam commentary given in section II. Section IV presents explanatory notes employing modern mathematical notations for the purposes of elucidation.

Figure 1: The title page and the first and the last folios of the manuscript A. Courtesy: Oriental Research Institute and Manuscript Library, University of Kerala.

Figure 2: The beginning and the ending pages of manuscript A, the content of which has been edited and presented in Section II of the article. Courtesy: Oriental Research Institute and Manuscript Library, University of Kerala.

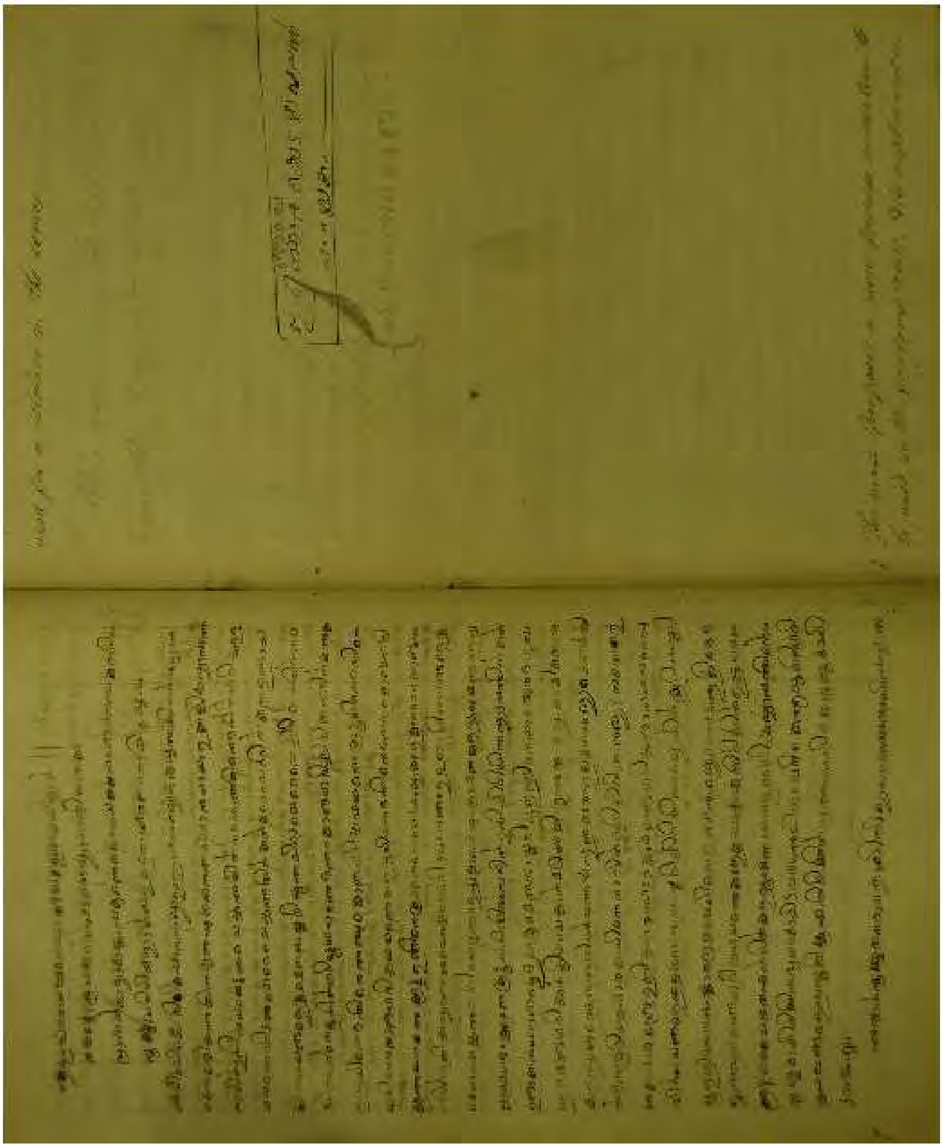

Figure 3a: A couple of folios of the manuscript B among the Whish collection available at Royal Asiatic Society, London.

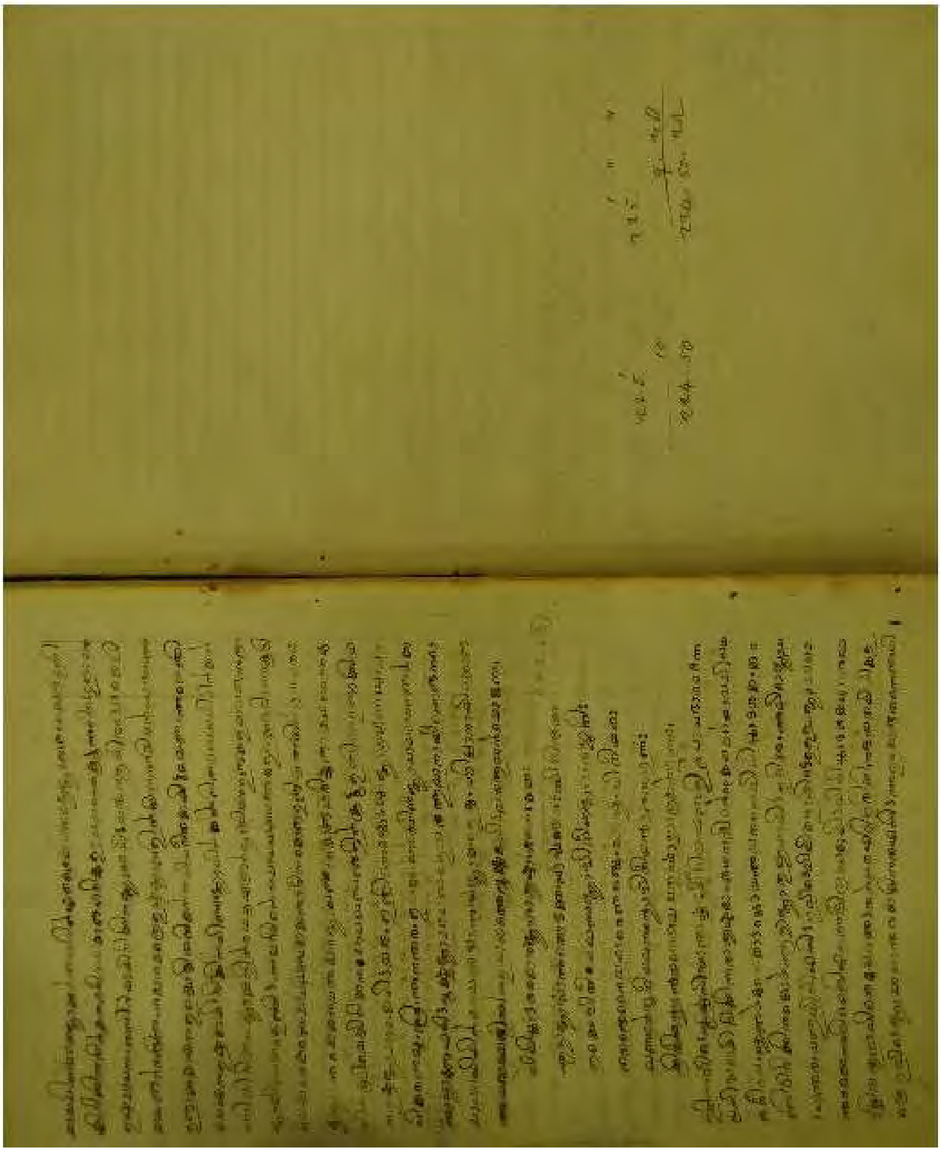

Figure 3b: A couple of folios of the manuscript B among the Whish collection available at Royal Asiatic Society, London.

2 The Text

2.1 Computation of tabular Rsines (Method I)

viliptādaśakonā jyā rāśyaṣṭāṃśa[11] dhanuḥkalāḥ ||
ādyajyārdhāttato[12] bhakte[13] sārdhadevāśvi[14] bhistataḥ |
tyakte dvitīyakhaṇḍajyā dvitīyā jyā ca tadyutiḥ[15] ||
tatastenaiva hāreṇa labdhaṃ śodhyaṃ dvitīyataḥ |
khaṇḍāttṛtīyakhaṇḍajyā[16] dvitīyastadyuto[17] guṇaḥ ||
tṛtīyaḥ syāttata[18] ścaivaṃ[19] caturthādyāḥ kramād[20] guṇāḥ ||[21]

അവിടെ സൂര്യസിദ്ധാന്താര്യഭടാദിഗ്രന്ഥങ്ങളിൽ ചാപഭാഗമെന്ന[22] പ്രസിദ്ധമായിരിക്കുന്ന രാശ്യഷ്ടമാംശധനുസ്സിന്റെ കലകൾ യാവചിലവ ശരീരസംഖ്യങ്ങൾ മുമ്പേ കാട്ടപ്പെട്ടവ. അവ തന്നെ വിലിപ്താദശകം കുറെഞ്ഞിട്ടിരിക്കുന്നപ്പോൾ ആദ്യജ്യാതുല്യയായിട്ടിരിക്കും.[23] അവിടെ ജ്യാചാപാന്തരം[24] പത്തു വിലി. എന്നിട്ട് ഇവിടെ വിലികളായിട്ടേ ഉള്ള ജ്യാചാപാന്തരത്തെ അറിയേണ്ടി എന്നു അഭിപ്രായമായിട്ടു വിലിപ്താദശകം എന്നു ചൊല്ലി. വസ്തുതഃ പിന്നെ മുപ്പത്തെട്ടു[25] തൽപര ഏറി ഇരിക്കുന്ന ഒമ്പതു വിലികളത്രേ അവിടെ ജ്യാചാപാന്തരമാകുന്നത്. എന്നിട്ട് അതുകൊണ്ടു തന്നെ അവിടെ ഹാരകവും 'നീലോബാലാരി'[26]

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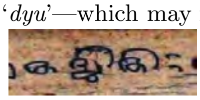
[11] A. ***rāśyaṃṣṭāṃśaḥ***; C. ***rāśyaṣṭāṃśaḥ***

[12] A. ***ādyajyārdhārttato***

[13] A. B. ***bhaktaṃ***; C. ***bhakteṃ***. The reading presented above is in accordance with the published work of *Tantrasaṅgraha* ([18] p.106), and we chose to replace it as the latter was found to be more appropriate.

[14] C. ***sārdhadevāgama***

[15] The unusual representation for the conjunct consonant '*dyu*'—which may not be possible to generate with the current software—found in Ms. A. is: 

[16] A. B. ***khaṇḍasyād***

[17] A. ***dyute***

[18] While all the three manuscripts A, B and C present the reading as ***syāstata*** we have modified it to ***syāttata*** which is in accordance with the edited version of the text *Tantrasaṅgraha* ([18] p.106). It may, however be pointed out here that in Ms. A, there is a variant reading presented just below the letter—that seems to have been done at a later stage by some other scribe—which is same as the correct version presented above.

[19] A. B. . . . ***caiva***

[20] A. B. ***kramāt***

[21] *Tantrasaṅgraha*, chapter II, verses 3b–6a; See [19], p. 17, or [18], p. 106.

[22] C. ചാപഭംഗമെന്ന

[23] C. തുല്യമായിട്ടിരിക്കും

[24] A. ജ്യാചാന്തരം

[25] C. മുമ്പെത്തൊട്ടു

[26] While specifying this number (in *kaṭapayādi* system), it is interesting to note that all the three manuscripts give variant readings. While A. and C. present the number 2323 30 (A. 'നീലോബാരിലാരി' and C. 'നീലോബാരലാരി'), B. presents the number 233 30 ('നീലോബാലാരി'). It is obvious from the context that the reading in B. is correct and hence that has been given above.

എന്നതിനും വിശേഷമുണ്ട്. 'രംഗേ ബാലാസ്ത്രീ' എന്നതു സൂക്ഷ്മമാകുന്ന ഹാരകം. എന്നാൽ പത്തു വിലി കുറഞ്ഞിരിക്കുന്ന[27] ശരീരത്തിങ്കേന്നു 'രംഗേ ബാലാസ്ത്രീ' എന്നതിനെക്കൊണ്ടു ഹരിച്ച ഫലം ഇലി യാതൊന്നു ഉണ്ടാകുന്നത്, അത് ആദ്യദ്വിതീയഖണ്ഡജ്യാന്തരമായിട്ടിരിക്കും. അതിനെ പിന്നെ നടേത്തെ ഖണ്ഡജ്യാവിങ്കന്നു കളെഞ്ഞാൽ രണ്ടാം ഖണ്ഡജ്യാവായിട്ടു വരും. അനന്തരം രണ്ടാം പിണ്ഡജ്യാവിങ്കൽ നിന്നു നടേത്തെ ഹാരകം കൊണ്ടു തന്നെ ഹരിച്ച ഫലം ഖണ്ഡജ്യാക്കളിൽ വച്ചു രണ്ടാമതിന്റേയും മൂന്നാമതിന്റേയും അന്തരമായിട്ടിരിക്കും. അതിനെപ്പിന്നെ രണ്ടാം ഖണ്ഡജ്യാവിങ്കന്നു കളഞ്ഞ ശേഷം മൂന്നാം ഖണ്ഡജ്യാവായിട്ടിരിക്കും. അതിന്റേയും രണ്ടാം പിണ്ഡജ്യാവിന്റെയും യോഗം മൂന്നാം പിണ്ഡജ്യാവയിട്ടു വരും. അനന്തരം അതിങ്കൽ നിന്നു ചൊല്ലിയവണ്ണം നാലാം ജ്യാവു[28] തുടങ്ങി ക്രമേണ ഉണ്ടാക്കാം. അതെങ്ങിനെ എന്നു മൂന്നാമതു തുടങ്ങി അതതു[29] പിണ്ഡജ്യാവിങ്കൽ നിന്നു[30] ആ ഹാരകം കൊണ്ടു തന്നെ ഹരിച്ച ഫലം അതിന്റെ മേലേ ഖണ്ഡജ്യാന്തരം അതിനെക്കളഞ്ഞു കീഴെ ഖണ്ഡജ്യാവു മേലേ ഖണ്ഡജ്യാവ് അതിനെ കൂട്ടിയ കീഴെ പിണ്ഡജ്യാവു മേലേ പിണ്ഡജ്യാവ് എന്നിങ്ങിനെ വൃത്തപാദം കഴിവോളവും ഖണ്ഡജ്യാന്തരഖണ്ഡജ്യാപിണ്ഡജ്യാക്കളെ എല്ലാറ്റേയും[31] ക്രമേണ ഉണ്ടാക്കിക്കൊള്ളൂ, എന്ന്.

2.2 Computation of Rsines (Method II)

അനന്തരം പഠിതജ്യാക്കളെ വരുത്തുന്നേടേത്തെ പ്രകാരാന്തരം—

vyāsārdhaṃ prathamaṃ nītvā tato vā'nyān guṇān nayet |[32]

അനന്തരം വ്യാസാർദ്ധത്തെ നടേ വരുത്തി അതിനെക്കൊണ്ടു മറ്റുള്ള ജ്യാക്കളെ എല്ലാറ്റെയും വരുത്തൂ[33] എന്നാകിലുമാം. അവിടെ വ്യാസാർദ്ധം എങ്ങിനെ വരുത്തേണ്ടൂ എന്നെടുത്തേക്കു ചൊല്ലുന്നു[34]—

sambandhaḥ[35] *niyamaścaivaṃ vijñeyo vyāsavṛttayoḥ* |

വ്യാസവും വൃത്തവും തങ്ങളിലുള്ള സംബന്ധം നിയമം എന്നു. ഇത്ര വ്യാസത്തിന്നു ഇത്ര വൃത്തം എന്നു താൻ വൃത്തത്തിന്നു വ്യാസം എന്നു താൻ ഉള്ളത് അതിന്റെ നിയമത്തെ മേലിൽ ചൊല്ലിയതിന്നു തക്കവണ്ണമറിഞ്ഞു കൊള്ളണം. അതിന്റെ പ്രകാരം —

[27] A. കുറഞ്ഞിരിക്കുന്ന രാ
[28] A. നാലാവു; B. നാലാജ്യാവു C. നാലാവതു
[29] C. അകത്തു
[30] A. നിന്നു ന്ന
[31] C. എല്ലാത്തേയും
[32] *Tantrasaṅgraha* chapter II, verse 6b; See [19], p. 18, or [18], p. 109.
[33] A. വരു
[34] A. ചൊല്ലുന്നൂ
[35] C. *sambandhaṃ*

2.3 Mādhava series for π

vyāse vāridhinihate rūpahṛte vyāsasāgarābhihate |
triśarādiviṣamasaṅkhyābhaktamṛṇaṃ svaṃ pṛthak kramāt kuryāt ||

അവിടെപ്പെരിക[36]സ്ഥാനസംഖ്യ ഉള്ളൊരു സംഖ്യെ ഒരു വൃത്തത്തിന്റെ വ്യാസമെന്നു കൽപിച്ചു[37] അതിന്നു വൃത്തം എത്ര എന്നറിയുവാനുപായം[38]. ഇക്കൽപിച്ച[39] വ്യാസത്തെ നാലിൽ ഗുണിച്ചു പലെടത്തു വച്ചു അതിൽ നടേത്തതിനെ ഒന്നിൽ ഹരിച്ചുള്ള ഫലത്തെ ഒരിടത്തു[40] വേറെ വച്ചു അതിങ്കൽ മറ്റേവറ്റിങ്കന്നുണ്ടാകുന്ന[41] ഫലങ്ങളെ ക്രമേണ സംസ്കരിപ്പൂ. അവിടെ വേറെ വച്ചിരിക്കുന്നതിൽ നടേത്തതിനെ മൂന്നിൽ ഹരിപ്പൂ. ഫലത്തെ ഒന്നു കൊണ്ടു ഹരിച്ചിരിക്കുന്നതിങ്കന്നു കളയൂ.[42] പിന്നെ ഒന്നിനെ അഞ്ചുകൊണ്ടു ഹരിച്ചിട്ടു[43] ഫലത്തെ കൂട്ടൂ. പിന്നെ ഒന്നിനെ ഏഴിൽ ഹരിച്ചിട്ടു ഫലത്തെക്കളയൂ.[44] പിന്നെ ഒന്നിനെ ഒമ്പതുകൊണ്ടു ഹരിച്ചുണ്ടായഫലത്തെ[45] കൂട്ടൂ[46]. ഇങ്ങിനെ ക്രമേണ ഒറ്റപ്പെട്ട സംഖ്യകളെ എല്ലാറ്റെയും[47] കൊണ്ടു നാലിൽ ഗുണിച്ച വ്യാസത്തെ വെവ്വേറെ ഹരിച്ച ഫലങ്ങളെ ഉണ്ടാക്കി ഫലങ്ങളെ ക്രമേണ വച്ചു അവറ്റിലോജങ്ങളെല്ലാറ്റെയും ക്രമേണ കളയൂ.[48] യുഗ്മങ്ങളെല്ലാറ്റെയും ക്രമേണ കൂട്ടൂ. ഇങ്ങിനെ[49] നാലിൽ ഗുണിച്ചിരിക്കുന്ന വ്യാസത്തിങ്കൽ ഉണ്ടാക്കുന്ന ഫലങ്ങളെയൊക്കെ[50] ക്രമേണ സംസ്കരിപ്പൂ.

2.4 *Antyasaṃskāra*: Estimating Mādhava series by applying end-correction

yatsaṅkhyayā'tra haraṇe kṛte nivṛttā hṛtistu jāmitayā |
tasyā ūrdhvagatāyā samasaṅkhyā[51] *taddalaṃ guṇo'nte syāt ||*

tadvargo[52] *rūpayuto hāro vyāsābdhighātataḥ prāgvat |*

[36] B. അവിടവിടെപെരിക
[37] A. B. കൽപിച്ച
[38] A. എന്നറിവാനുപായം
[39] A. ഇക്കൽപിച്ചെ
[40] A. ഒരീടത്തു
[41] B. മറ്റെവറ്റിൽകന്ന് ഉണ്ടായ
[42] A. കളെയൂ; B. കളവൂ
[43] B. ഹരിച്ച
[44] A. ഏഴി ഹരിച്ചിട്ട ഫലത്തെക്കളെയൂ; B. ഏഴിനെക്കൊണ്ട ഹരിച്ച ഫലത്തെ ക്കളവൂ
[45] A. ഹരിച്ചു ഫലത്തെ; B. ഹരിച്ച അ ഫലത്തെ
[46] A. ക്കൂട്ടൂ
[47] A. adds "ക്രമേണ കളെയൂ". The parenthesis (not clearly observed) introduced in the manuscript perhaps indicates that this string is spurious, and does not belong to the original text.
[48] A. കളെയൂ; B. കളവൂ
[49] B. എന്നിങ്ങനെ
[50] A. ഫലങ്ങളെയൊക്ക
[51] A. B. *ssamasaṅkhyā*
[52] A. *tadvaggo*

tābhyāmāptaṃ svamṛṇe[53]*kṛte dhane śodhanañca karaṇīyam* ||
sūkṣmaḥ paridhiḥ saḥsyāt bahukṛtvo haraṇato'tisūkṣmaśca ||[54]

എന്നിങ്ങനെ[55] യാതൊരോജസംഖ്യ കൊണ്ടു ഹരിച്ചാൽ[56] ക്രിയ ചെയ്‌വാൻ മതിയാകുന്നു,[57] അസ്സംഖ്യേടെ മേത്തേ യുഗ്മസംഖ്യ യാതൊന്നു അതിന്റെ അർദ്ധം ഗുണകാരം. ഈ[58] യുഗ്മസംഖ്യേടെ വർഗത്തിൽ ഒന്നു കൂട്ടിയത് ഹാരകം. പിന്നെ നാലിൽ ഗുണിച്ചിരിക്കുന്ന വ്യാസത്തെ ഈ ഗുണകാരം[59] കൊണ്ടു ഗുണിച്ച്, ഈ ഹാരകം കൊണ്ടു ഹരിച്ചുണ്ടായ ഫലത്തെ മുമ്പിൽ ഫലങ്ങളെല്ലാറ്റെയും സംസ്കരിച്ചിരിക്കുന്ന സംഖ്യയിൽ കൂട്ടി സംസ്കരിപ്പൂ.

ഇതിന്റെ സംസ്കാരപ്രകാരം പിന്നെ. ഈ യുഗ്മസംഖ്യേടെ അടുത്തു കീഴെ ഓജസംഖ്യ[60] കൊണ്ടു ഹരിച്ച ഫലത്തെക്കളക ചെയ്തത് എന്നിരിക്കിൽ ഇഫ്ഫലത്തെ കൂട്ടൂ. കൂട്ടുക ചെയ്തത് എന്നിരിക്കിൽ കളയൂ.[61]

എന്നാലത് ഇക്കൽപിച്ച വ്യാസത്തിന്റെ സൂക്ഷ്മമായിട്ടിരിക്കുന്ന[62] വൃത്തമായിട്ടു വരും. അവിടെപ്പെരികെ ആവൃത്തി ഹരിച്ച ഫലങ്ങളെ സംസ്കരിപ്പൂ. എന്നിരിക്കിൽ വൃത്തം പെരിക സൂക്ഷ്മമായിട്ടു വരും.

2.4.1 More accurate correction-term

asmāt[63] *sūkṣmataro'nyo vilikhyate kaścanāpi*[64] *saṃskāraḥ* ||[65]

ഇവിടെ ഒടുക്കത്തു ചൊല്ലിയ സംസ്കാരഫലത്തെ ഇതിനേക്കാൾ സൂക്ഷ്മമായിട്ടു ചൊല്ലുന്നു.

ante samasaṅkhyādalavargaḥ saiko guṇaḥ sa eva[66] *punaḥ* |
yugaguṇito rūpayutaḥ[67] *samasaṃkhyādalahato bhaved hāraḥ* ||

[53] A. *svamṛṇe na*

[54] A slightly variant reading is found in the citation of the same set of verses by Śankara in his *Kriyākramakarī* ([8] p. 379): . . . *kṛte dhane kṣepa eva karaṇīyaḥ* | *labdhaḥ paridhiḥ sūkṣmaḥ bahukṛtvo haraṇato'tisūkṣmaḥ syāt* ||

[55] A. എന്നിങ്ങിനെ; B. ഇങ്ങിനെ

[56] A. ഹരിച്ചൽ

[57] A. B. മടിയാകുന്നു

[58] A. ഇ

[59] A. ഇഗ്ഗുണകാരം

[60] A. ഒജസംഖ്യ

[61] A. കളെയൂ

[62] A. സൂക്ഷ്മമായീട്ടിരിക്കുന്ന

[63] In *Yuktidīpikā* we find '*ebhyaḥ*' instead of '*asmāt*' (see [18] p. 103).

[64] C. *kaścannavi*

[65] A. C. add '*iti*' at the end of the verse.

[66] A. B. present the combined form '*samasaṅkhyādalavargassaiko guṇassa eva*'

[67] A. *rūpahataḥ*; B. C. as well as other published works such as *Yuktidīpikā* (see [18] p. 103) present the reading given above. That the reading given by A. is wrong is also evident from the computations involved.

triśarādiviṣamasaṅkhyāharaṇāt parametadeva vā kāryam[68] ||[69]

ഒടുക്കത്തു[70] ഹരിച്ച ഓജസംഖ്യേടെ[71] മീത്തേ യുഗ്മസംഖ്യേടെ [72] അർദ്ധത്തെ വർഗ്ഗിച്ച് ഒന്നു കൂട്ടിയതു ഗുണകാരമാകുന്നത്. ഇഗ്ഗുണകാരത്തെ[73] തന്നെ പിന്നെ[74] നാലിൽ ഗുണിച്ച് ഒന്നു കൂട്ടി അതിനെപ്പിന്നെ ഇസ്സമസംഖ്യാർദ്ധം കൊണ്ടും ഗുണിപ്പൂ.[75] അത് ഹാരമാകുന്നത്.[76]

ഇവിടെ മൂന്നു അഞ്ചു തുടങ്ങിയുള്ള വിഷമസംഖ്യകൊണ്ടും ഹരിച്ച[77] ഫലങ്ങളെ സംസ്കരിച്ചു അനന്തരം[78] ഇച്ചൊല്ലിയ ഗുണകാരം കൊണ്ടു ചതുർഘ്ന വ്യാസത്തെ ഗുണിച്ച് ഈ ഹാരകം കൊണ്ടു ഹരിച്ച ഫലങ്ങളേയും സംസ്കരിപ്പൂ. എന്നാലിപ്പരിധി മുമ്പിച്ചൊല്ലിയതിനെക്കാൾ ഏറ്റവും സൂക്ഷ്മമായിട്ടിരിക്കും. ഇപ്പക്ഷത്തിങ്കൽ മുമ്പിച്ചൊല്ലിയ സംസ്കാരത്തെ ചെയ്കയും വേണ്ട.

2.5 Circumference by an alternative method

പിന്നെ പ്രകാരാന്തരം—

vyāsavargādravihatāt[79] *padaṃ syāt prathamaṃ phalam* |
tatastattatphalāccāpi[80] *yāvadicchan tribhirharet*[81] ||
rūpādyayugmasaṅkhyābhir labdheṣveṣu[82] *yathākramaṃ* |

[68]The reading found in B. as well as *Yuktidīpikā* (see [18] p. 103) is given above. A. and C. present the reading '*parametadeva kāryaḥ*' which is grammatically wrong.
[69]A. C. add '*iti*' at the end of the verse.
[70]A. ഒടുക്കത്ത
[71]A. ഒജസംഖ്യേടെ
[72]A. C. om. മീത്തേ യുഗ്മസംഖ്യേടെ
[73]B. ഈ ഗുണകാരത്തെ
[74]A. C. വി
[75]A. ഗുണിച്ചപ്പൂ
[76]C. Last two sentences from ഇഗ്ഗുണകാരത്തെ... to ... ഹാരകമാകുന്നത് are repeated in the Ms. C.
[77]Here, all the three Mss. A, B and C, present the reading ഗുണിച്ച, which means "multiplying". However, this does not seem to be appropriate as what is involved in the series is division by odd numbers 3, 5, etc. and not multiplication. Hence we have corrected the reading to ഹരിച്ച (dividing) instead of ഗുണിച്ച.
[78]A. സംസ്കരിച്ചനന്തരം
[79]A. *vyāsasyavargād rāśihatāt*; C. *vyāsasyavargād ravihatāt*. Both these readings are evidently faulty as they do not satisfy the rules of *anuṣṭubh* metre in which the verse has been composed. Moreover, the reading *rāśi* in A. can also be misleading, since it has been often used in the literature to represent number 30—*rāśi*, by definition being is a 30° division along the ecliptic.
[80]A. *"mastattatphalāccāpi" tatastattatphalāccāpi*. The string inside the quotation mark appears out of context, and seems to have been introduced by the scribe by mistake.
[81]C. *tribhiharet*
[82]A. ... *billabdheṣveṣu*

viṣamāṇāṃ[83] *yutestyakte*[84] *yugmayoge vṛtirbhavet*[85] ||[86]

പിന്നെ ഒരു[87] വലിയൊരു[88] സംഖ്യേ വ്യാസമെന്നു കൽപിച്ചു അതിനെ വർഗ്ഗിച്ചു പ ന്ത്രണ്ടിൽ ഗുണിച്ചു മൂലിപ്പൂ. അതു നടേത്തെ രാശിയാകുന്നത്. അതിനെ വേറെ വച്ചു മൂന്നിൽ[89] ഹരിച്ച ഫലം രണ്ടാം രാശിയാകുന്നത്. അതിനെ വേറെ വച്ച് മൂന്നിൽ[90] ഹരിച്ച ഫലം മൂന്നാം രാശിയാകുന്നത്. പിന്നെയും ഈവണ്ണം തന്നെ അതതു ഫലത്തെ വേറെ വച്ചു പ്പിന്നത്തെ പ്പിന്നത്തെ ഫലങ്ങളെ[91] ഉണ്ടാക്കു വേണ്ടുവോളം. പിന്നെ ഈ ഫലങ്ങളെ[92] ക്രമേണ[93] വെച്ച നടേത്തതിനെ ഒന്നു കൊണ്ടു ഹരിപ്പൂ. രണ്ടാമതിനെ[94] മൂന്നു കൊണ്ടു ഹരിപ്പൂ. മൂന്നാമതിനെ അഞ്ചു കൊണ്ടു ഹരിപ്പൂ. നാലാമതിനെ ഏഴു കൊണ്ടു ഹരിപ്പൂ.

ഇങ്ങിനെ[95] പിന്നെപ്പിന്നെ ഫലങ്ങളെ മേലെ മേലെ ഓജസംഖ്യകളെക്കൊണ്ട് ഹരിച്ചു ണ്ടായ ഫലങ്ങളേ എല്ലാറ്റേയും[96] ക്രമേണ വെച്ചാൽ, അവറ്റിൽ നടേത്തത് മൂന്നാമത് അ ഞ്ചാമത് എന്നിയൊറ്റപ്പെട്ട ഫലങ്ങളെ ഒക്കെ വേറെ എടുത്ത് തങ്ങളിൽ കൂട്ടൂ. പിന്നെ രണ്ടാമത് നാലാമത് എന്നിത്യാദി യുഗ്മഫലങ്ങളെ വേറെ എടുത്ത തങ്ങളിൽ കൂട്ടി, ഓജഫല യോഗത്തിങ്കന്ന് യുഗ്മഫലയോഗത്തെക്കളഞ്ഞശേഷം കൽപിച്ച വ്യാസത്തിന്റെ വൃത്തമായിട്ട് ഇരിക്കുന്നു.

2.6 Transformed Mādhava series: I

samapañcāhatayo yā rūpādyayujāṃ[97] *caturghnamūlayutāḥ* |

[83] B. *viṣamānāṃ*

[84] A. B. *tyakte*; C. *tyakto*

[85] A. B. add *iti* at the end of the verse

[86] The same set of verses appear in *Kriyākramakarī* with significant variation in the reading:
vyāsavargād ravihatāt padaṃ syāt prathamaṃ phalam |
tadāditastrisaṅkhyaaptaṃ phalaṃ syāduttarottaram ||
rūpādyayugmasaṅkhyābhir hṛteṣveṣu yathākramam |
viśamāṇāṃ yutestyaktavā samā hi paridhirbhavet || (see [8] p. 387).

[87] A. ഒര

[88] A. വലിയൊര

[89] A. മുമ്പിൽ

[90] A. മുമ്പിൽ

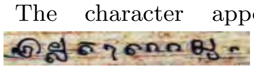
[91] A. ഫലത്തെ

[92] A. ഇഫ്ഫലങ്ങളെ (പിന്നെ is omitted); C. om. ഈ

[93] The text from ക്രമേണ... to the ... നാലിൽ which appears in the fifth line of the commentary of the verse *samapañcāhata* ... appearing in the next section is missing in Ms. C.

[94] C. രണ്ടാമത്തേതിനെ

[95] C. ഇങ്ങനെ

[96] A. The character appearing after 'ല്ല' with the bar above in the image  is perhaps a scribal error, and is not to be found in other manuscripts.

[97] A. B. *rūpādyayujāḥ*. Since this reading seems to be grammatically flawed, the one presented above is in accordance with the other edited works of K. V. Sarma (see [8] p. 390, [18] p. 102, [6] p. 80).

tābhiḥ ṣoḍaśaguṇitād vyāsāt pṛthagāhṛtetu[98] *viṣamayuteḥ* |
samaphalayoge tyakte syādiṣṭavyāsasambhavaḥ paridhiḥ ||

ഒന്ന്, മൂന്ന്, അഞ്ച്[99] എന്നു തുടങ്ങിയുള്ള ഓജസംഖ്യകളുടെ സമപഞ്ചാഹതികൾ ആകുന്നതു യാവ[100] ചിലവ അവറ്റിൽ മൂലത്തെ നാലിൽ ഗുണിച്ചു കൂട്ടൂ. ഇവ ക്രമേണ ഹാരകങ്ങളാകുന്നവ. അവിടെ സമപഞ്ചാഹതി എന്ന[101] മൂന്നിനെ വർഗ്ഗിച്ച് അതിനെത്തന്നെ പിന്നെയുമൊരിക്കൽ[102] വർഗ്ഗിച്ച് പിന്നെ അതിനെത്തന്നെ പിന്നെയും[103] മൂന്നിൽ ഗുണിച്ചത് മൂന്നിന്റെ സമപഞ്ചാഹതിയാകുന്നത്. അതിൽ, മൂന്നിനെ നാലിൽ ഗുണിച്ചു കൂട്ടൂ. അത് മൂന്നിന്റെ സമപഞ്ചാഹതി ചതുർഘ്നമൂലയുതമായിട്ടിരിക്കുന്നത്.[104] പിന്നെ ഈവണ്ണം തന്നെ അഞ്ചിന്റെ വർഗ്ഗവർഗ്ഗത്തെ[105] അഞ്ചിൽ ഗുണിച്ചത്, അഞ്ചിന്റെ സമപഞ്ചാഹതി. ഇതിൽ[106] അഞ്ചിനെ നാലിൽ ഗുണിച്ചു കൂട്ടിയത് പിന്നത്തെ ഹാരകമാകുന്നത്.

ഇങ്ങിനെ ക്രമേണ ഹാരകങ്ങളെ ആവോളം ഉണ്ടാക്കി പതിനാറിൽ ഗുണിച്ചിരിക്കുന്ന വ്യാസത്തെ അത്രേടത്തു വച്ച് ക്രമെണ ഓരോ ഹാരകം കൊണ്ടു ഓരോന്നിനെ ഹരിച്ച ഫലങ്ങളെ ക്രമെണ ഉണ്ടാക്കൂ. അവറ്റിൽ ഓജങ്ങളുടെ യോഗത്തിങ്കേന്നു യുഗ്മങ്ങളുടെ യോഗത്തെക്കളഞ്ഞ് ശേഷം കൽപിച്ച വ്യാസത്തിന്റെ വൃത്തമായിട്ടിരിക്കും.

2.7 Transformed Mādhava series: II

vyāsādvāridhinihatāt pṛthagāptaṃ tryādyayugvimūlaghanaiḥ |
trighnavyāse svamṛṇaṃ kramaśaḥ kṛtvāpi paridhi[107]*rāneyaḥ* ||

അനന്തരം വ്യാസത്തെ നാലിൽ ഗുണിച്ചതിനെപ്പലേടത്തു വച്ചു[108] ക്രമേണ ഹരിപ്പൂ. അവിടെത്തേതിനെ മൂന്നിന്റെ ഘനത്തിൽ മൂന്നു[109] കളഞ്ഞ് ശേഷം കൊണ്ടു ഹരിപ്പൂ. പിന്നെ രണ്ടാമതിനെ അഞ്ചിന്റെ ഘനത്തിൽ അഞ്ചു കളഞ്ഞു ശേഷം കൊണ്ടു ഹരിപ്പൂ. ഇങ്ങിനെ[110] മേലെ മേലെ ഓജസംഖ്യാഘനങ്ങളിൽ നിന്നു തന്റെ തന്റെ മൂലം കളഞ്ഞ് ശേഷം കൊണ്ടു ഹരിപ്പൂ. ഫലങ്ങളെ ക്രമേണ വെപ്പൂ. പിന്നെ വ്യാസത്തെ മൂന്നിൽ ഗുണിച്ചതിന്റെ നടേത്തെ

[98] A. *pṛthagāhatetu*; This reading is erroreous as what is to be done is division and not multiplication. A variant reading *pṛthagāhṛteṣu* found in Ms. B. as also elsewhere (see [8] p. 390) confirms that the reading presented in the text above is correct.
[99] A. om. 'അ' in അഞ്ച്
[100] A. യവ
[101] A. സമപഞ്ചാഹതികൾ ആകുന്നതു എന്ന
[102] B. പിന്നെയുമൊരിക്കും
[103] A. പിന്നെയുമതിനെ instead of പിന്നെ അതിനെത്തന്നെ പിന്നെയും
[104] A. C. ചതുർഘ്നമൂലമായിട്ടിരിക്കുന്നത്
[105] A. C. അഞ്ചിന്റെ വർഗ്ഗത്തെ
[106] A. C. ഇതി
[107] A. *kṛtvā peridhi*
[108] A. B. വെച്ച
[109] A. C. മൂന്നിന്നു
[110] C. ഇങ്ങനെ

ഫലത്തെ കൂട്ടൂ. രണ്ടാം ഫലത്തെ കളയൂ.[111] പിന്നെ മൂന്നാം ഫലത്തെ കൂട്ടൂ. ഇങ്ങിനെ[112] ഒന്നിടയുള്ള[113] ഫലങ്ങളെ ക്രമേണ കൂട്ടുകയും കളെകയും ചെയ്‌വൂ. എന്നാലത് പരിധിയായിട്ടു വരും.

2.8 Transformed Mādhava series: III

dvyādiyujāṃ vā kṛtayaḥ vyekā hārā dvinighnaviṣkambhe |
dhanamṛṇamante'ntyordhvagataujakṛtirdvisahitā[114] *haro dvighnī*[115] ||

പിന്നെ രണ്ടു നാലെന്നു തുടങ്ങിയുള്ള യുഗ്മസംഖ്യകളുടെ വർഗ്ഗങ്ങളിൽ നിന്നു ഓരോന്നു കളഞ്ഞത് ഹാരകമാകുന്നത്.[116] അവറ്റെക്കൊണ്ടു നാലിൽ ഗുണിച്ചു വ്യാസത്തിങ്കേന്നു വെവ്വേറെ ഹരിപ്പൂ. ഉണ്ടായ ഫലങ്ങളെ ക്രമേണ ഇരട്ടിച്ച വ്യാസത്തിങ്കേന്നു കൂട്ടുകയും കളെകയും ചെയ്‌വൂ ഒന്നിടയിട്ടു.[117] പിന്നെ ഒടുക്കത്തെ യുഗ്മസംഖ്യേടെ മേലേ ഓജസംഖ്യേടെ വർഗ്ഗത്തിൽ രണ്ടു കൂട്ടി അതിനെ ഇരട്ടിച്ചത് സംസ്കാരഹാരകമാകുന്നത്. അതിനെക്കൊണ്ട് ചതുർഘ്നവ്യാസത്തെ ഹരിച്ച ഫലത്തെയും സംസ്കരിപ്പൂ. അതു പരിധിയായിട്ടിരിക്കും.

2.9 Transformed Mādhava series: IV

dvyādeścaturādervā caturadhikānāṃ nirekavargāssyuḥ |
hārāḥ kuñjara[118]*guṇito viṣkambhaḥ svamatikalpito*[119] *bhājyaḥ* |
phalayutirekatra[120] *vṛtirbhājyadalaṃ phalavihīnamanyatra* ||

രണ്ടാദിയായി അതിൽ നന്നാലേറി ആറ്, പത്ത്, പതിന്നാല് എന്നു[121] തുടങ്ങിയുള്ള സംഖ്യകളുടെ വർഗ്ഗങ്ങളിൽ ഓരോന്നു കളഞ്ഞശേഷങ്ങൾ ഹാരകങ്ങളാകുന്നത്. അവറ്റെക്കൊണ്ടു ക്രമേണ എട്ടിൽ ഗുണിച്ച വ്യാസത്തെ ഹരിപ്പൂ. അവിടെയുണ്ടായ ഫലങ്ങളെ ഒക്കെ[122] ത

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[111] A. ക്കളെയൂ; B. കളവൂ
[112] C. ഇങ്ങനെ
[113] A. ഒന്നിടയിട ഉള്ള
[114] The reading found in all the Mss. A, B and C, as well as in *Kriyākramakarī* ([8] p. 390) is '*kṛtidvisahitā*'. However, it is grammatically incorrect if the *visarga* is missing in *kṛtiḥ*—which actually takes the form of '*r*' due to *sandhi* rules. Hence, we have included the correct form in the main text by introducing *visarga*.
[115] C. *dvinighni*. There is a variant reading in *Kriyākramakarī* ([8] p. 390), where we find '*harasyārdhaṃ*' in the place of '*haro dvighnī*'.
[116] B. ഹാരകങ്ങളാക്കുന്നത്
[117] C. ചെയ്‌വൂ. ഒന്നിടയിട്ടു പിന്നെ
[118] A. *kubdhara*
[119] A. *kalpyato*
[120] A. B. '*phalayutirādye vṛttaṃ*'; C. '*phalayutirārekatra*'. Since both these readings do not convey proper sense, we had a suspision that all the manuscripts could be presenting distorted readings at this point, and hence consulted the published work *Kriyākramakarī* ([8] p. 390). The reading found there seemed to be more appropriate and hence the same has been included in the text above.
[121] A. എന്നിതു; C. എന്നിത്
[122] A. B. ഒക്ക

ങ്ങളിൽ കൂട്ടൂ. അത് കൽപിതവ്യാസത്തിന്റെ വൃത്തമായിട്ടു വരും. പിന്നെ നാലു തുടങ്ങി നന്നാലേറി എട്ട്, പന്ത്രണ്ട് എന്നിത്യാദി സംഖ്യകളുടെ വർഗ്ഗങ്ങളിൽ ഓരോന്നു കളഞ്ഞത് ഹാരകങ്ങളാകുന്നത്. അവറ്റെക്കൊണ്ട് എട്ടിൽ ഗുണിച്ച വ്യാസത്തെ വെവ്വേറെ ഹരിച്ച് ഉണ്ടായ ഫലങ്ങൾ തങ്ങളിൽ കൂട്ടൂ. ചതുർഘ്നവ്യാസത്തിങ്കന്നു കളയൂ.[123] ശേഷം വൃത്തമായിട്ടിരിക്കും.

2.10 Concluding the discussion on Mādhava series

ityevaṃ mahato vyāsāt mahāntaṃ paridhinnayet[124] |
tato'lpamanupātena nītvā paṭhatu tadyathā ||

ഇങ്ങിനെ വലിയൊരു വ്യാസത്തെ കൽപിച്ചു അതിനെക്കൊണ്ടു വലിയൊരു പരിധിയെ[125] ഉണ്ടാക്കൂ. പിന്നെ ത്രൈരാശികം ചെതീട്ടൂ ചെറിയോ ചില വ്യാസവൃത്തങ്ങളെ വരുത്തിപ്പഠിച്ചുകൊള്ളൂ. അതിങ്ങിനെ[126]—

iṣṭavyāse hate nāgavedavahnyabdhikhendubhiḥ |
tithyaśvivibudhairbhakte sūkṣmaḥ paridhirbhavet ||
paridhervyatyayāccaivaṃ[127] *susūkṣmaṃ vyāsamānayet* ||

തനിക്കിഷ്ടമായിരിപ്പൊരു[128] വ്യാസത്തെ 'ദേവലോവിനയീ'[129] എന്നതിനെക്കൊണ്ടു ഗുണിച്ചു 'മാന്യസ്ത്രീബാല'[130] എന്നുകൊണ്ട് ഹരിച്ച ഫലം തന്റെ ഇഷ്ടവ്യാസത്തിന്റെ സൂക്ഷ്മമായിരിക്കുന്ന പരിധിയായിട്ടിരിക്കും.[131] പിന്നെ അതിങ്കന്നു[132] വിപരീതമായിട്ടു ഇഷ്ടപരിധിയുടെ വ്യാസത്തെയും വരുത്താം. അവിടെ[133] തനിക്കിഷ്ടമായിരിക്കുന്ന പരിധിയെ 'മാന്യസ്ത്രീബാല' എന്നതിനെക്കൊണ്ടു ഗുണിച്ച് 'ദേവലോവിനയീ' എന്നതിനെക്കൊണ്ട്[134] ഹരിച്ച ഫലം ഇഷ്ടപരിധിയുടെ സൂക്ഷ്മമാകുന്ന വ്യാസമായിട്ടു വരും.

[123] A. കളെയൂ; B. കളവൂ
[124] C. *paridhirnayet*
[125] C. പരിധി
[126] B. അതിങ്ങിനെ എന്ന
[127] A. *paridhevyatyayāccaivaṃ*
[128] B. തനിക്ക ഇഷ്ടമായി ഇരിപ്പൊര
[129] C. ദേവല്ലോവിനയീയം
[130] A. മാന്യസ്ത്രീ ര ക ര ബല ബാല
[131] B. പരിധിയായീട്ട വരും
[132] B. ഇതിൻകന്നു
[133] A. അവിടെ അവിടെ
[134] The string 'ഗുണിച്ച് 'ദേവലോവിനയീ' എന്നതിനെക്കൊണ്ട്' is missing in C.

3 Translation of the text

3.1 Computation of tabular Rsines (Method I)

> The *jyā* of one-eighth of the arc corresponding to a *rāśi* (expressed) in minutes, is $10''$ short of that (length of the arc in minutes). The quantity obtained by dividing the first Rsine (*jyārdha*) by $233\frac{1}{2}$, and subtracting it from the same, is the second *khaṇḍajyā*. This added to it (the first *jyā*) is the second *jyā*. The result obtained by dividing that (the second *jyā*) by the same divisor ($233\frac{1}{2}$) is to be subtracted from the second *khaṇḍajyā*. This is the third *khaṇḍajyā*. This added to it is the third *guṇa* (Rsine). From that, fourth *guṇa* etc., have to be obtained in order.

In *Sūryasiddhānta*, *Āryabhaṭīya*, etc., it is well known that the term *cāpabhāga* (arc-bit) is used to refer to one-eighth of a *rāśi* which is [equal to] $225'$ (*śarīra*). The same ($225'$), when diminished by $10''$ (*viliptādaśaka*), would be equal to the first Rsine. There, the difference between the arc and the corresponding Rsine is $10''$. [It must be understood that] here it is stated to be ten *viliptās*, [only] with the intention of specifying the difference between the Rsine and the arc in terms of seconds (*viliptās*). Actually it is only thirty-eight thirds (*tatpaās*) in excess of nine seconds (*vilis*) that makes the difference between the Rsine and the corresponding arc. Thus, there is a distinction between the [actual] divisor and $233'30''$ (*nīlobālāri*). [In fact,] the accurate divisor is $233'32''$ (*raṅgebālāstrī*). Therefore, the result obtained in seconds by dividing $225'$ diminished by $10''$ by $233'32''$ (*raṅgebālāstrī*), would be the difference of the first and second *khaṇḍajyā*-s. If that is subtracted from the first *khaṇḍajyā*, then second *khaṇḍajyā* would be obtained. [Adding this to the first, second *piṇḍajyā* is obtained.] Then the result obtained by dividing second *piṇḍajyā* by the previous divisor ($233'32''$) would be the difference of the second and third *khaṇḍajyā*-s. Now, this [result] if subtracted from the second *khaṇḍajyā*, would be the third *khaṇḍajyā*. The sum of this [result] and the second *piṇḍajyā* would be the third *piṇḍajyā*. Herefrom, the fourth and other Rsines can be obtained as said [earlier]. This is how it is [done]. By dividing further *piṇḍajyās*—starting from the third—by that *hāraka*, one gets the next difference of *khaṇḍajyā*; subtracting that from previous *khaṇḍajyā* the next *khaṇḍajyā* will be obtained. That added to the previous *piṇḍajyā* gives the succeeding *piṇḍajyā*. In this way, the difference of *khaṇḍajyā*-s, [and,] therefrom the *khaṇḍajyā*-s and *piṇḍajyās* may be obtained till the end of the quadrant of the circle.

3.2 Computations of Rsines (Method II)

Here is another method for obtaining the *paṭhitajyā*-s—

> Having obtained the radius (*vyāsārdha*) first, the other Rsines may be obtained from that.

Now, all the Rsines can be found by first obtaining the radius (*vyāsārdha*). Here we explain, how the radius is to be obtained.

> The relation and invariability [of that relation] between the diameter and [the circumference of] the circle has to be understood.

[Here we describe] the invariable relation between the diameter and [the circumference of] a circle. The procedure for obtaining the circumference corresponding to a given diameter, and the diameter corresponding to a given circumference is to be understood from the following description. The procedure is [as follows].

3.3 Mādhava series for π

> To the diameter multiplied by four and divided by unity, the products of the diameter and four divided by the odd numbers like three, five, etc., have to be applied negatively and positively in order.

Here is a means to obtain [the circumference of] a circle whose diameter is considered to be a large number. The diameter is multiplied by four and kept separately in many places; [Among them] the first one is divided by one and the result should be stored separately. This is to be repeatedly corrected using the results obtained from the other ones. Among the others which are placed separately, divide the first one by three and the result should be subtracted from the one that was divided by one. Then, [to this] add the result obtained by dividing the next one by five. [From that] again, subtract the result obtained by dividing the next one by seven. Again, add the quantity obtained by dividing the next one by nine. Thus, having divided the product of diameter and four by all the odd numbers and placing the results sequentially, subtract all the odd ones in order. [Similarly] add all the even ones in order. This is the way in which the results obtained by dividing the product of four and the diameter [by all odd numbers] are to be applied.

3.4 *Antyasaṃskāra*: Estimating Mādhava series by applying end-correction

> Half of the succeeding even number, at whichever [odd] number the process of division is terminated, because of boredom [caused by the slow convergence] is the multiplier. The square of that [even number] added by unity is the divisor. The result obtained from these two multiplied by the product of the diameter and four as earlier [and] has to be added if the earlier term [in the series] has been subtracted; subtracted if the earlier term has been added. The resulting circumference is very accurate; in fact, more accurate than the one which may be obtained by continuing the division process [and summing up large number of terms in the series].

Having divided by some odd number, if one feels like terminating the process [due to boredom], then the half of the succeeding even number is to be taken as multiplier (*guṇakāra*). The square of this even number increased by one is the divisor (*hāraka*). The result obtained—by multiplying the product of four and the diameter by this *guṇakāra* and dividing by the *hāraka*—is to be applied as a correction-term to the sum of the results earlier.

Now, the procedure for applying the correction-term (*phala*) [is stated]. If the previous term ($\frac{1}{p}$)—where the divisor p is odd, and wherefrom was obtained the present even number [in the correction-term]—was subtractive, then add this result [to the series]. If that was additive, then subtract.

The value obtained thus would be the accurate [value of the circumference] of the circle of the considered diameter. The correction-term obtained must be applied to the partial sum [of the series] having done more number of divisions. Only then the circumference would be more accurate.

3.4.1 More accurate correction-term

> Another correction-term that is more accurate than this is being stated–

Here, another correction-term is being stated that is more accurate than the one stated earlier.

> Square of half of the even term added by unity is [taken to be] the multiplier. Again, the same multiplied by four, added by one and

> [the sum further] multiplied by half of the even number becomes the divisor. Instead of keeping on dividing by three, five, etc., it is better to apply this [correction-term].

The square of half of the odd number that happened to be the divisor when the series was terminated, added by 1 forms the multiplier (*guṇakāra*). Having multiplied this multiplier by 4, adding 1 to that, this has to be further multiplied by half of this even number. This becomes the divisor.

Having applied the results of dividing [four times the diameter] by the odd numbers three, five, etc., [to four times the diameter] the result [obtained by] multiplying the product of the diameter and four by the multiplier (*guṇakāra*) that is mentioned [before] and dividing by this divisor, is also applied. Then the circumference obtained would be far more accurate than the one stated earlier. In case of [applying] this, it is not necessary to do the previous correction (*saṃskāra*).

3.5 Circumference by an alternative method

Yet another method–

> The square-root of the product of the square of the diameter and 12 (*ravi*) is the first result (*prathamaphala*). Divide that, and all the successive *phalas* obtained [from the previous one] by three. [Keep generating *phalas*] as you desire (*yāvadicchan*). Then these [*phalas*] are divided, in order, by the odd numbers 1,3, etc. When the sum of the even ones is subtracted from that of the odd ones, the circumference is obtained.

Then, considering a very large number as diameter, and having squared that multiply [that] by twelve and take the square-root. That forms the first term (*rāśi*). By placing this (*rāśi*) separately and dividing by three, the second *rāśi* is obtained. Keeping this [second *rāśi*] separately and dividing by three the result [obtained] is the third *rāśi*. Like this, again placing those results separately keep generating successive terms, as many as desired. Then having kept these results in order, divide the first [*rāśi*] by one. Divide the second by three. Divide the third by five. Divide the fourth by seven.

All the results thus obtained by dividing the terms by the successive odd numbers have to be placed sequentially. Having picked up the odd terms, first, third, fifth, etc. separately from this [sequence], add them together. Then, [similarly] choosing the even terms second, fourth, etc. separately, add them together. The remainder obtained by subtracting the sum of the even-terms from the sum of the odd-terms would be [the circumference of] the circle of a given diameter.

3.6 Transformed Mādhava series: I

> The fifth-powers of odd numbers commencing from one are added to four times themselves. By dividing the product of sixteen and the diameter by these separately, and subtracting the sum of the even terms from the odd ones we get the circumference corresponding to a given diameter.

The fifth-powers of the odd numbers such as one, three, five, etc. are added to their bases multiplied by four. They form the successive divisors. Here, having squared three, [and] again squaring that [square] and again multiplying that by three, the fifth-power (*samapañcāhati*) of three would be obtained. Three multiplied by four should be added to that. That gives the sum of the *samapañcāhati* of three and the base (three) multiplied by four. In the same way, the square of the square of five multiplied by five is *samapañcāhati* of five; that added to the product of four and five, forms the next divisor (*hāraka*).

In this way, having obtained several divisors sequentially, divide the product of diameter and sixteen kept in several places by them, and store the results obtained by dividing each of them with these divisors. Among them, the remainder obtained by subtracting the sum of even terms from the sum of odd terms would be the circumference of given diameter.

3.7 Transformed Mādhava series: II

> The diameter multiplied by four is divided separately by the cubes of the odd numbers starting with three diminished by their bases. By applying this positively and negatively, in order, to the diameter multiplied by three, the circumference should be obtained.

Thereafter, having placed the product of the diameter and four in several places, divide [them] sequentially by the remainder [obtained] by subtracting three from the cube of three. [Proceeding] like this, divide them by the remainders obtained by subtracting the cubes of successive odd numbers by themselves. Place the results sequentially. Now, add the first term to the product of diameter and three. Subtract the second term. Then add the third term. In this way, the terms have to be added to and subtracted from alternatively. Thus, the circumference is obtained.

3.8 Transformed Mādhava series: III

> Or, [dividing the product of four and the diameter by] the squares of the even numbers starting with two, diminished by one are considered as divisors [of the terms] which are added to and subtracted from twice the diameter [alternatively]. The square of the odd number that succeeds the last even number, added by two and doubled would be the [correction] divisor.

Thereafter, the squares of the even numbers like two, four, etc. subtracted by one form the divisors. Divide the product of the diameter and four by these [divisors] separately. The results obtained have to be added to and subtracted from twice the diameter alternatively. Then, adding two to the square of the odd number which succeeds the last even number, and doubling it, the correction-divisor (*saṃskārahāraka*) is obtained. The result obtained by dividing the product of diameter and four by this [divisor] should also be applied. That will be the circumference.

3.9 Transformed Mādhava series: IV

> The squares of the numbers starting from two or four and increased by four, are diminished by unity. If these [numbers obtained sequentially] are taken to be the divisors (*hāras*) then the desired diameter multiplied by eight will be dividend (*bhājya*). [Then] the circumference would be the sum of the results at one place [where it is started with the square of two] and half of the *bhājya* diminished by [sum of] all the terms at the other place.

The squares of the numbers, beginning with two and increasing by four like six, ten, fourteen, etc., diminished by one form the divisors. Eight times the diameter should be divided successively by these [divisors]. Add the results obtained together. This will be [the circumference of] the circle of a given diameter. Now, the squares of numbers beginning with four and increasing by four like eight, twelve, etc., diminished by one form the divisors. The results obtained by dividing eight times the diameter separately by these [divisors] should be added together. Subtract [this sum] from four times the diameter. The remainder will be [the circumference of] the circle.

3.10 Concluding the discussion on Mādhava series

> Like this, [one should] calculate the large circumference corresponding to a large diameter. Small [circumferences] could be obtained by using the rule of proportion (*anupāta*). That is [done] in this way.

Like this, obtain the large circumference by considering a large diameter. Then obtain smaller diameter and the circumference using the rule of three. That is [done] in this way.

> The desired diameter multiplied by 104348 (*nāgavedavahnyabdhikhendu*) and divided by 33215 (*tithyaśvivibudha*) will give the accurate circumference. Like this, in the reverse way, accurate diameter could be calculated from the circumference.

The result obtained by multiplying the desired diameter by 104348 and divided by the 33215 would be the circumference of the desired diameter. Then reversing the operation, the diameter corresponding to a circumference can also be calculated. There, the result obtained by multiplying the desired circumference by 33215 and dividing by 104348 would be the accurate diameter of the desired circumference.

4 Mathematical notes

4.1 Computation of tabular Rsines (Method I)

Here we explain the procedure for finding the accurate values of the 24 Rsines (*jyās*) as described in *Tantrasaṅgraha*,[135] with the help of Figure 4. As shown in the figure, the quadrant is divided into 24 equal parts, each part P_iP_{i+1} ($i = 0, 1, \dots, 23$) corresponding to $225'$. Before proceeding further, we need to introduce a few terminologies—namely *khaṇḍajyā*, *khaṇḍajyāntara* and *piṇḍajyā*—that would be employed by us in our discussion. With reference to Figure 4, they are defined as follows:

$$\begin{aligned} \textit{piṇḍajyā} &= P_iN_i = J_i & i &= 1, 2, \dots, 24, \\ \textit{khaṇḍajyā} &= P_{i+1}N_{i+1} - P_iN_i = \Delta_i & i &= 1, 2, \dots, 23, \\ \textit{khaṇḍajyāntara} &= \Delta_i - \Delta_{i+1} & i &= 1, 2, \dots, 23. \end{aligned} \tag{1}$$

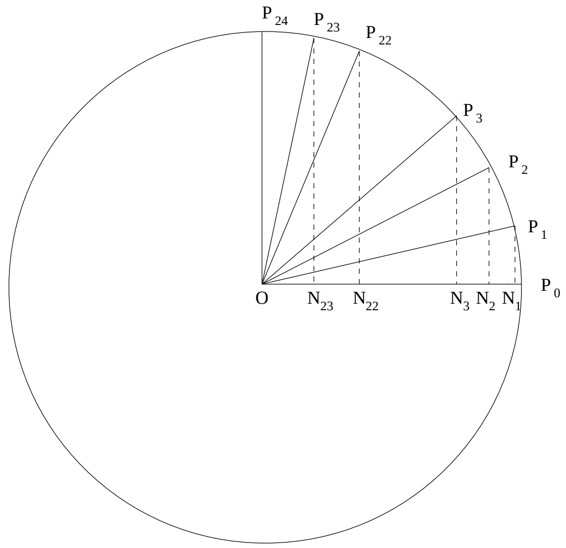


Figure 4: Determination of the *jyā* corresponding to the arc-lengths which are multiples of $225'$.

The term *piṇḍajyā* essentially refers to the whole or the tabulated *jyā*. They are 24 in number, represented by $J_1, J_2 \dots J_{24}$ and are expressed in minutes of arc. The last *piṇḍajyā*, namely $P_{24}N_{24} = P_{24}O$ is referred to as *trijyā*, and its length is equal to the radius of the circle. The difference between the successive *piṇḍajyās* are referred to as the *khaṇḍajyā*-s. The verses commencing with '*viliptādaśakonā*', after specifying the value of the first *piṇḍajyā* proceeds to describe the procedure for generating the successive *piṇḍajyās* from the previous ones.

The length of the first *piṇḍajyā* (P_1N_1) is stated to be one-eighth of a *rāśi* expressed in minutes minus 10 seconds, [which] is equal to $224'\,50''$. This is also equal to the first *khaṇḍajyā*. Thus we have

$$jy\bar{a}\ P_0P_1 = P_1N_1 = J_1 = 224'\,50'' = \Delta_1. \tag{2}$$

This can be understood as follows. In Figure 4,

$$P_0\hat{O}P_1 = \frac{90}{24} = 3.75^\circ = 225' = 0.65949846 \text{ radian.} \tag{3}$$

While the first *piṇḍajyā* is taken to be $225'$ in some of the texts like *Āryabhaṭīya* and *Sūryasiddhānta*, here it is taken to be $224'50''$. It seems the choice of this

[135]The procedure described here is essentially the same as in *Āryabhaṭīya*, but for the values of the first *jyā* (which is taken to be $224'50''$ instead of $225'$) and the divisor (which is taken to be $233\frac{1}{2}$ instead of $225'$).

value is based on the relation:[136]

$$\sin\alpha \approx \alpha - \frac{\alpha^3}{3!}. \tag{5}$$

The rationale behind the procedure given in the text for obtaining the successive *jyās*, from the first one may be outlined as follows. The second *khaṇḍajyā*, Δ_2, is defined as,

$$\begin{aligned}\Delta_2 &= J_2 - J_1 \\ &= R(\sin 2\alpha - \sin\alpha),\end{aligned} \tag{6}$$

where, $P\hat{O}P_2 = 2\alpha$. Since, $\sin 2\alpha = 2\sin\alpha\cos\alpha$, we have

$$\Delta_2 = R\sin\alpha(2\cos\alpha - 1). \tag{7}$$

Rewriting the above equation,

$$\Delta_2 = R\sin\alpha[1 - 2(1-\cos\alpha)]. \tag{8}$$

Now, using the fact $R\sin\alpha = J_1 = \Delta_1$, we have

$$\begin{aligned}\Delta_1 - \Delta_2 &= \Delta_1.2(1-\cos\alpha) \\ &= \frac{\Delta_1}{233\frac{1}{2}} = \frac{224'50''}{233\frac{1}{2}},\end{aligned} \tag{9}$$

Since $2(1-\cos\alpha) \approx 0.004282153 \approx \frac{1}{233\frac{1}{2}}$ for $\alpha = 225'$. Again from (9), we get second *khaṇḍajyā* as

$$\begin{aligned}\Delta_2 &= \Delta_1 - \frac{J_1}{233\frac{1}{2}} \\ &\approx 224'50'' - 57.77'' \\ &\approx 223'52''.\end{aligned} \tag{10}$$

The second *piṇḍajyā* is given by

$$\begin{aligned}J_2 &= J_1 + \Delta_2 \\ &= 224'50'' + 223'52'' \\ &= 448'42''.\end{aligned} \tag{11}$$

[136]In fact, this approximation to sine function is explicitly given in the II chapter of *Tantrasaṅgraha* (see verse 17, beginning with *śiṣṭacāpaghanaṣaṣṭhabhāgataḥ*). Thus,

$$P_1N_1 = R\sin\alpha \approx \frac{21600}{2\pi}\left(\alpha - \frac{\alpha^3}{6}\right) = 224.8389' \approx 224'50''. \tag{4}$$

In general, the i^{th} *khaṇḍajyā* is given by

$$\Delta_i = \Delta_{i-1} - \frac{J_{i-1}}{233\frac{1}{2}}, \tag{12}$$

and the i^{th} *piṇḍajyā* by

$$J_i = J_{i-1} + \Delta_i. \tag{13}$$

4.2 Computation of Rsines (Method II)

This method of finding Rsines involves the value of *trijyā* which is nothing but the last Rsine (J_{24}). That is, $R = R\sin 90 = J_{24}$. Since the value of first *jyā* (J_1) is already known, with these two *jyās* (the last and the first), the text describes the method to find the value of the penultimate *jyā* (J_{23}).

$$\begin{aligned} \sqrt{J_{24}^2 - J_1^2} &= \sqrt{(R\sin 24\alpha)^2 - (R\sin\alpha)^2} \\ &= \sqrt{R^2 - (R\sin\alpha)^2} \\ &= R\cos\alpha \\ &= R\sin(24\alpha - \alpha) \\ &= R\sin 23\alpha \ (J_{23}). \end{aligned} \tag{14}$$

Now that J_{23} is obtained, the text defines a *gu.na* and *hāra* as follows:

$$\begin{aligned} gu.na &= 2(R - R\sin 23\alpha) \\ h\bar{a}ra &= R \end{aligned} \tag{15}$$

Using them a recursion relation is formulated,[137] making use of which all the tabular difference of *khaṇḍajyā*-s and hence the values of the 24 *jyās* can be obtained. Since this method of obtaining Rsines requires a precise value of *trijyā* (R)—as may be noted from (14) and (15)— it is said in the text: "find the radius first and from that obtain the other Rsines".

4.3 Mādhava series for π

The series encoded in the verse "*vyāse vāridhi...*" is the well known series

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7}\ldots, \tag{16a}$$

[137]For more details, the readers may consult the Explanatory Notes on *Tantrasaṅgraha* by K. Ramasubramanian and M. S. Sriram. See [20].

which may also be written as

$$C = 4D\left(1-\frac{1}{3}+\frac{1}{5}-\dots\right) , \tag{16b}$$

where C is the circumference of the circle whose diameter is D. We shall now present the derivation of the above series as outlined in *Yuktibhāṣā* [6, pp. 183–98] . For this, let us consider the quadrant OP_0P_nS of the square circumscribing the given circle (see Figure 1) of radius r. Divide the side P_0P_n into n equal parts (n very large). The resulting segments P_0P_i's ($i = 1, 2, \dots, n$) are known as the *bhujās* and the line joining its tip and the centre OP_i's are known as *karṇas*. The points of intersection of these *karṇas* and the circle are denoted by A_i. The *bhujās* P_0P_i, the *karṇas* k_i and the east-west line OP_0 form right-angled triangles whose hypotenuses are given by

$$k_i^2 = r^2 + \left(\frac{ir}{n}\right)^2 . \tag{17}$$

Considering two successive *karṇas*—ith and the previous one as shown in the figure—and the pairs of similar triangles, $OP_{i-1}C_i$ and $OA_{i-1}B_i$ and $P_{i-1}C_iP_i$ and P_0OP_i, it can be shown that

$$A_{i-1}B_i = \left(\frac{r}{n}\right)\left(\frac{r^2}{k_{i-1}k_i}\right). \tag{18}$$

Now the text presents the crucial argument: When n is large, the Rsines $A_{i-1}B_i$ corresponding to different arc-bits $A_{i-1}A_i$ can be taken as the arc-bits themselves. Thus, $\frac{1}{8}$th of the circumference of the circle can be written as the sum of the contributions given by (18).

$$\frac{C}{8} \approx \left(\frac{r}{n}\right)\left[\left(\frac{r^2}{k_0k_1}\right)+\left(\frac{r^2}{k_1k_2}\right)+\dots+\left(\frac{r^2}{k_{n-1}k_n}\right)\right]. \tag{19}$$

It is further argued in the text that the denominators $k_{i-1}k_i$ may be replaced by the square of either of the *karṇas* i.e., by k_{i-1}^2 or k_i^2 since the difference is negligible. Thus (19) may be re-written in the form

$$\begin{aligned}
\frac{C}{8} &= \sum_{i=1}^{n}\frac{r}{n}\left(\frac{r^2}{k_i^2}\right) \\
&= \sum_{i=1}^{n}\left(\frac{r}{n}\right)\left(\frac{r^2}{r^2+\left(\frac{ir}{n}\right)^2}\right) \\
&= \sum_{i=1}^{n}\left[\frac{r}{n}-\frac{r}{n}\left(\frac{\left(\frac{ir}{n}\right)^2}{r^2}\right)+\frac{r}{n}\left(\frac{\left(\frac{ir}{n}\right)^2}{r^2}\right)^2-\dots\right]
\end{aligned} \tag{20}$$

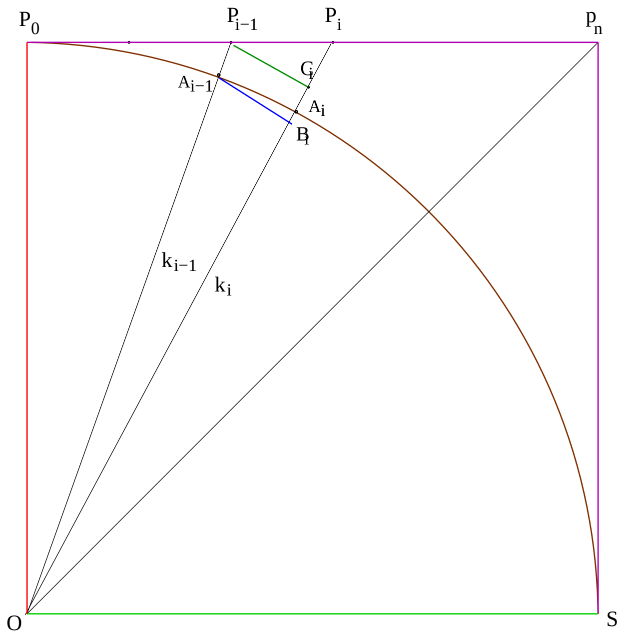


Figure 5: Geometrical construction used in the proof of the infinite series for π.

In the series expression for the circumference given above, factoring out the powers of $\frac{r}{n}$, the summations involved are that of even powers of the natural numbers. It was known to the Kerala mathematicians [6, p. 196] that for large n

$$\sum_{i=1}^{n} i^k \;\approx\; \frac{n^{k+1}}{k+1}\,. \tag{21}$$

Using the above relation in (20), we arrive at the result[138]

$$\frac{C}{8} = r\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right), \tag{22}$$

which is same as $(16b)$, the well known series for $\frac{\pi}{4}$.

[138]In modern terminology, the above derivation amounts to the evaluation of the following integral

$$\frac{C}{8} = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{r}{n}\right) \left(\frac{r^2}{r^2 + \left(\frac{ir}{n}\right)^2}\right) = r\int_0^1 \frac{dx}{1+x^2}.$$

4.4 *Antyasaṃskāra*: Estimating Mādhava series by applying end-correction

It is well known that the Mādhava series given above (16*b*) is an extremely slowly converging series—so much so that it requires millions of terms to be considered even to get an accuracy upto four decimal places. This problem has been brilliantly solved by Mādhava by applying a certain technique, which may be called *antyasaṃskāra*, in which the series is rewritten in the following form:

$$C \approx 4D\left(S_p + \frac{1}{a_p}\right). \tag{23}$$

Here S_p represents the sum of the terms in the series that is terminated at $\frac{1}{p}$ (p being an odd number of our choice) and the term $\frac{1}{a_p}$ is a rational approximation to the remaining terms in the series. The application of this term is known as *antyasaṃskāra*. In the discussion that follows this term will simply be referred to as end-correction. The nomenclature stems from the fact that a correction (*saṃskāra*) is applied towards the end (*anta*) of the series, when it is terminated after a certain number of terms.

Three successive approximations to the end-correction ($\frac{1}{a_p}$) have been given by Mādhava, and they may be expressed as:

$$\frac{1}{a_p(1)} = \frac{1}{2(p+1)} \tag{24}$$

$$\frac{1}{a_p(2)} = \frac{\left(\frac{p+1}{2}\right)}{(p+1)^2+1} \tag{25}$$

$$\frac{1}{a_p(3)} = \frac{\left(\frac{p+1}{2}\right)^2+1}{\left[\left(\left(\frac{p+1}{2}\right)^2+1\right)4+1\right]\left(\frac{p+1}{2}\right)}, \tag{26}$$

where as mentioned earlier, p represents the last odd number used as the divisor, at which the series was terminated. The end-correction represented by (25).

The series obtained by applying the end-correction may be written as

$$C \approx 4D\left(1 - \frac{1}{3} + \frac{1}{5} - \ldots + (-1)^{\frac{p-1}{2}}\frac{1}{p} + (-1)^{\frac{p+1}{2}}\frac{\left(\frac{p+1}{2}\right)}{(p+1)^2+1}\right) \tag{27}$$

We now proceed to explain the rationale behind the end-correction given in (27). Suppose we terminate the series given by (16*b*) after the term $\frac{1}{p}$, where

p is an odd number of our choice, and consider applying the end-correction (*antyasaṃskāra*) $\frac{1}{a_p}$, as a substitute of the remaining terms in the series, then the series becomes

$$C \approx 4D\left(1-\frac{1}{3}+\frac{1}{5}\ldots+(-1)^{\frac{p-1}{2}}\frac{1}{p}+(-1)^{\frac{p+1}{2}}\frac{1}{a_p}\right) \tag{28}$$

Now the question arises as to what gives the necessary licence to terminate the series at $\frac{1}{p}$ and represent the rest of the infinite term by a single term (*antyasaṃskāra*)? And more importantly, what is the guarantee that the end-correction gives the right estimate of the remainder terms in the series?

The argument adduced in both *Yuktibhāṣā* and *Kriyākramakarī* in favor of terminating the series at any desired term, still ensuring the accuracy, may be summarized as follows. Let the series for C be written as

$$C = 4D\left(1-\frac{1}{3}+\frac{1}{5}\ldots+(-1)^{\frac{p-3}{2}}\frac{1}{p-2}+(-1)^{\frac{p-1}{2}}\frac{1}{a_{p-2}}\right), \tag{29}$$

where $\frac{1}{a_{p-2}}$ is the end-correction applied after odd denominator $p-2$. On the other hand, if the end-correction $\frac{1}{a_p}$, is applied after the odd denominator p, then

$$C = 4D\left(1-\frac{1}{3}+\frac{1}{5}\ldots+(-1)^{\frac{p-1}{2}}\frac{1}{p}+(-1)^{\frac{p+1}{2}}\frac{1}{a_p}\right). \tag{30}$$

If the end-corrections chosen were exact, viz., they happen to be the right estimate of the remainder terms in the series, then both the series (29) and (30) should yield the same result. That is,

$$\frac{1}{a_{p-2}} = \frac{1}{p}-\frac{1}{a_p} \qquad \text{or} \qquad \frac{1}{a_{p-2}}+\frac{1}{a_p}=\frac{1}{p}, \tag{31}$$

is the criterion that must be satisfied for the end-correction (*antyasaṃskāra*) to lead to the exact result.

The criterion given by (31) is trivially satisfied when we choose $a_{p-2} = a_p = 2p$. However, this value $2p$ cannot be assigned to both the correction-divisors[139] a_{p-2} and a_p because both the corrections should follow the same rule. That is,

$$\begin{aligned} \text{if} \quad a_{p-2} = 2p \quad &\Rightarrow \quad a_p = 2(p+2) \\ \text{or, if} \quad a_p = 2p \quad &\Rightarrow \quad a_{p-2} = 2(p-2). \end{aligned}$$

[139] By the term correction-divisor (*saṃskāra-hāraka*), the divisor of the end-correction is meant.

We can, however, have both a_{p-2} and a_p as close to $2p$ as possible. The choice of $a_{p-2} = 2p - 2$ and $a_p = 2p + 2$ would fulfill the above criteria. It may also be noted that there will persist a difference of 4 between a_{p-2} and a_p since $p - 2$ and p are doubled. Hence, the first (order) estimate of the correction-divisor is given as, "double the even number above the last odd-number divisor p",

$$a_p = 2(p + 1). \tag{32}$$

But, it can be seen right away that, with this value of the correction-divisor, the condition for accuracy stated above in (31) is not exactly satisfied. Hence a measure of inaccuracy or error called *sthaulya* $E(p)$ is introduced.

$$E(p) = \left[\frac{1}{a_{p-2}} + \frac{1}{a_p}\right] - \frac{1}{p}\ . \tag{33}$$

Since this error cannot be eliminated, the objective is to find the correction-divisors a_p such that the inaccuracy $E(p)$ is minimized.

When we set $a_p = 2(p + 1)$, the inaccuracy will be

$$\begin{aligned} E(p) &= \left[\frac{1}{(2p-2)} + \frac{1}{(2p+2)}\right] - \frac{1}{p} \\ &= \frac{1}{(p^3 - p)}\ . \end{aligned} \tag{34}$$

This estimate of the inaccuracy, E_p being positive, shows that the correction has been overdone and hence there has to be a reduction in the magnitude of the end-correction chosen. This means that the correction-divisor has to be increased. If we take $a_p = 2p + 3$, thereby leading to $a_{p-2} = 2p - 1$, we have

$$\begin{aligned} E(p) &= \left[\frac{1}{(2p-1)} + \frac{1}{(2p+3)}\right] - \frac{1}{p} \\ &= \frac{(-2p + 3)}{(4p^3 + 4p^2 - 3p)}\ . \end{aligned} \tag{35}$$

Now, the inaccuracy happens to be negative. But, more importantly, it has a term proportional to p in the numerator. Hence, for large p, $E(p)$ given by (25) varies inversely as p^2, while for the divisor given by (22), $E(p)$ as given by (24) varied inversely as p^3. In fact, it can be shown that among all possible correction divisors of the type $a_p = 2p + m$, where m is an integer, the choice of $m = 2$ is optimal, as in all other cases there will arise a term proportional to p in the numerator of the inaccuracy $E(p)$.

From (24) and (25) it is obvious that, if we want to reduce the inaccuracy and thereby obtain a better correction, then a number less than 1 has to be added to the correction-divisor (22) given above. If we try adding *rūpa* (unity) divided by the correction divisor itself, i.e., if we set $a_p = 2p + 2 + \frac{1}{(2p+2)}$, the contributions from the correction-divisors get multiplied essentially by $\left(\frac{1}{2p}\right)$. Hence, to get rid of the higher order contributions, we need an extra factor of 4, which will be achieved if we take the correction divisor to be

$$a_p = (2p+2) + \frac{4}{(2p+2)} = \frac{(2p+2)^2 + 4}{(2p+2)} \,. \tag{36}$$

Then, correspondingly, we have

$$a_{p-2} = (2p-2) + \frac{4}{(2p-2)} = \frac{(2p-2)^2 + 4}{(2p-2)} \,. \tag{37}$$

We can then calculate the inaccuracy to be

$$\begin{aligned} E(p) &= \left[\frac{1}{(2p-2) + \frac{4}{2p-2}} + \frac{1}{(2p+2) + \frac{4}{2p+2}} \right] - \left(\frac{1}{p}\right) \\ &= \left[\frac{(4p^3)}{(4p^4+16)} \right] - \frac{(16p^4+64)}{4p(4p^4+16)} \\ &= \frac{-4}{(p^5+4p)} \,. \end{aligned} \tag{38}$$

Clearly, the *sthaulya* with this (second order) correction divisor has improved considerably, in that it is now proportional to the inverse fifth power of the odd number.[140]

At this stage, we may display the result obtained for the circumference with the correction term as follows. If only the first order correction (22) is employed, then we will have

$$C \approx 4D \left[1 - \frac{1}{3} + \ldots + (-1)^{\frac{(p-1)}{2}} \frac{1}{p} + (-1)^{\frac{(p+1)}{2}} \frac{1}{(2p+2)} \right] . \tag{39}$$

If the second order correction (23) is taken into account, we have

$$C \approx 4D \left[1 - \frac{1}{3} + \ldots + (-1)^{\frac{(p-1)}{2}} \frac{1}{p} + (-1)^{\frac{(p+1)}{2}} \frac{1}{(2p+2) + \frac{4}{(2p+2)}} \right]$$

[140]It may be noted that if we take any other correction-divisor $a_p = 2p + 2 + \frac{m}{(2p+2)}$, where m is an integer, we will end up having a contribution proportional to p^2 in the numerator of the inaccuracy $E(p)$, unless $m = 4$. Thus the above form (26) is the optimal second order choice for the correction-divisor.

$$\approx \quad 4D\left[1-\frac{1}{3}+\ldots+(-1)^{\frac{(p-1)}{2}}\frac{1}{p}+(-1)^{\frac{(p+1)}{2}}\frac{\frac{(p+1)}{2}}{(p+1)^2+1}\right]. \tag{40}$$

The verse due to Mādhava presenting the infinite series for $\frac{\pi}{4}$ cited earlier is in fact, the first of a group of four verses that present the series along with the above end-correction. The other verse presenting the end-correction represented by (26) is given by [8], p. 390.

ante samasaṅkhyādalavargaḥ saiko guṇaḥ ...

Incorporating the end-correction given by the above verse, Mādhava series takes the form

$$C \approx 4D\left[1-\frac{1}{3}+\ldots+(-1)^{\frac{(p-1)}{2}}\frac{1}{p}+(-1)^{\frac{(p+1)}{2}}\frac{\left(\frac{p+1}{2}\right)^2+1}{\left[\left(\left(\frac{p+1}{2}\right)^2+1\right)4+1\right]\left(\frac{p+1}{2}\right)}\right]. \tag{41}$$

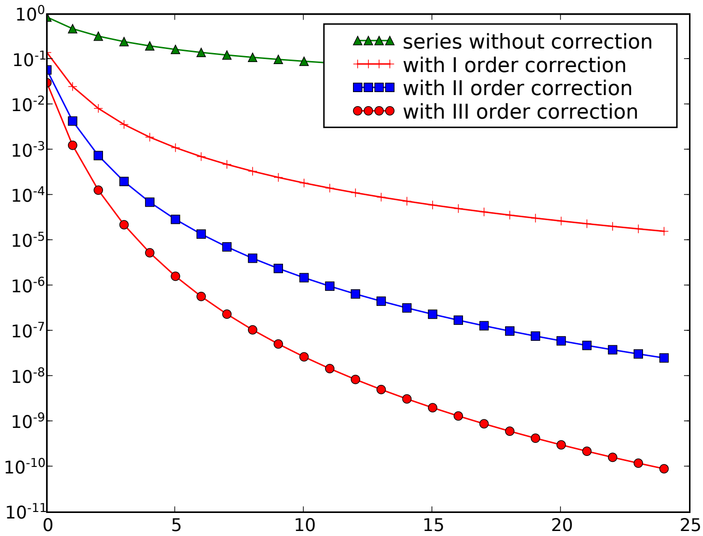
Graph depicting the variation of error in the estimate of π using the three successive end-corrections given by (24)–(26) by truncating the series at different values of p, is shown in Figure 6. For the purpose of convenience, in the figure, we have referred to the plots corresponding to the three successive end-correction terms as first, second and third order respectively. It may be noted that, when we use the III order end-correction, by just considering about 25 terms in the series, we are able to obtain π value correct to 10 decimal places. For more detailed treatment on the topic the readers are referred to *Yuktibhāṣā* [6, pp. 201-207].

Figure 6: Graph depicting the accuracy that is obtained in making an estimate of the value of π using the Mādhava series by truncating it at different values of p and employing the three successive end-corrections given by (24)–(26). Here x-axis represents the number of terms considered in the series which is $\frac{p+1}{2}$.

4.5 Circumference by an alternative method

Among the various forms in which the circumference of a circle has been expressed by Mādhava, the following verses present it in a specific form, the rationale behind which—as will be shown below—can be easily understood (by those acquainted with modern mathematics) using the Taylor series expansion of the arc-tangent function. The series given by Mādhava is

$$C=\sqrt{12D^2}-\frac{\sqrt{12D^2}}{3.3}+\frac{\sqrt{12D^2}}{3^2.5}-\frac{\sqrt{12D^2}}{3^3.7}+\cdots, \tag{42}$$



The procedure for obtaining the circumference given in the verse *vyāsa-vargāt*... may be outlined as follows. Here, the text prescribes the procedure to find several *phalas* P_1, P_2, P_3 and so on upto P_n where n is an integer of our choice (*yāvadicchan*). The successive *phalas* are to be obtained from the preceding *phalas* by dividing by three, viz., $P_{i+1} = \frac{P_i}{3}$, with the first *phala*, P_1, defined as

$$\begin{aligned} P_1 &= \sqrt{ravi \times vy\bar{a}savarga} \\ &= \sqrt{12D^2}, \end{aligned} \tag{43}$$

where D is the diameter of the circle. It is further said that the *phalas* P_i $(i = 1, 2, \dots n)$ thus obtained have to be divided by the odd numbers 1,3,5, etc. sequentially. That is,

$$P_{i+1} = \frac{P_i}{3.(2i+1)} \qquad (i = 1, 2, 3 \dots). \tag{44}$$

Now, the circumference is said to be obtained by subtracting the sum of the odd *phalas* from that of the even ones. In other words, if C is the circumference, then

$$\begin{aligned} C &= \sum_{i=1}^{n} (-1)^{i+1} P_i \\ &= \sqrt{12D^2} - \frac{\sqrt{12D^2}}{3.3} + \frac{\sqrt{12D^2}}{3^2.5} - \cdots. \end{aligned} \tag{45}$$

The rationale behind the above expression can be understood as follows. It is well known that the Taylor expansion of $\frac{1}{1+x^2}$ is

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \dots \tag{46}$$

Since the derivative of $\tan^{-1} x = \frac{1}{1+x^2}$, by integrating the above equation we get the Taylor series expansion of $\tan^{-1} x$. That is,

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots \tag{47}$$

By making the substitution $\theta = tan^{-1}x$, we have $x = \tan\theta = \frac{\sin\theta}{\cos\theta}$. Hence, the above equation reduces to the form

$$\begin{aligned} \theta &= \left(\frac{\sin\theta}{\cos\theta}\right) - \frac{1}{3}\left(\frac{\sin\theta}{\cos\theta}\right)^3 + \frac{1}{5}\left(\frac{\sin\theta}{\cos\theta}\right)^5 - \dots \\ \text{or} \quad R\theta &= \frac{R\sin\theta}{1\cos\theta} - \frac{R\sin\theta}{3\cos\theta}\left(\frac{\sin\theta}{\cos\theta}\right)^2 + \frac{R\sin\theta}{5\cos\theta}\left(\frac{\sin\theta}{\cos\theta}\right)^4 - \dots \end{aligned} \tag{48}$$

If we take the arc-length equal to one-twelfth of the circumference, i.e., ($R\theta = \frac{C}{12}$), which is equivalent to taking $\theta = \frac{\pi}{6}$, then $\frac{\sin\theta}{\cos\theta} = \frac{1}{\sqrt{3}}$. Substituting this in the above series we get

$$\begin{aligned} \frac{C}{12} &= R\left(\frac{1}{\sqrt{3}}\right) - \frac{R}{3}\left(\frac{1}{3\sqrt{3}}\right) + \frac{R}{5}\left(\frac{1}{3^2\sqrt{3}}\right) \\ \frac{C}{12} &= \left(\frac{R}{\sqrt{3}}\right)\left[1 - \frac{1}{3.3} + \frac{1}{5.3^2} - \frac{1}{7.3^3}\cdots\right] \\ C &= 12\left(\frac{D}{\sqrt{12}}\right)\left[1 - \frac{1}{3.3} + \frac{1}{3^2.5} - \frac{1}{3^3.7}\cdots\right] \\ &= \sqrt{12D^2}\left(1 - \frac{1}{3.3} + \frac{1}{3^2.5} - \frac{1}{3^3.7}\cdots\right), \end{aligned} \tag{49}$$

which is the same as the series given by (45).

4.6 Transformed Mādhava series: I

Consider the expressions for the circumference of a circle given by (16b) and (49) in terms of its diameter. It may be noted that in the former series only the successive odd numbers appear in the denominator with first power, whereas, in the latter, we find successive powers of 3 (3^i, $i = 0, 1, 2, \dots$) multiplying the odd numbers appearing in the denominator. Obviously the latter series will be converging faster and hence would be fetching far more accurate results by considering fewer terms in the series.

Besides this series, which of course is unquestionably a calculus masterpiece, Mādhava seems to have given many interesting fast convergent approximations of his series by incorporating the end-corrections given by (24)–(26) into the series itself.

The expression for the circumference of a circle presented in the verses commencing with *samapañcāhatayo* may be written as

$$
\begin{aligned}
C &= \frac{16D}{1^5+4.1} - \frac{16D}{3^5+4.3} + \frac{16D}{5^5+4.5} - \cdots \\
&= 16D\left(\frac{1}{1^5+4.1} - \frac{1}{3^5+4.3} + \frac{1}{5^5+4.5} - \cdots\right) \qquad (50)
\end{aligned}
$$

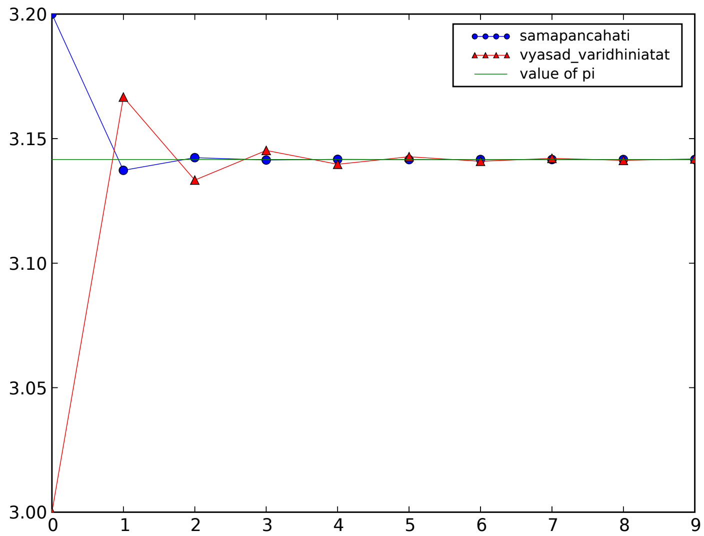
The rationale behind the above expression can be understood with the help of equations (28) and (33) giving the expression for circumference and the expression for inaccuracy of *sthaulya* respectively. Using them, the circumference may be expressed in terms of *sthaulyas* as follows:

$$
\begin{aligned}
C &= 4D\left[\left(1-\frac{1}{a_1}\right) + \left(\frac{1}{a_1}+\frac{1}{a_3}-\frac{1}{3}\right) - \left(\frac{1}{a_3}+\frac{1}{a_5}-\frac{1}{5}\right) - \cdots\right] \\
&= 4D\left[\left(1-\frac{1}{a_1}\right) + E(3) - E(5) + E(7) - \cdots\right], \qquad (51)
\end{aligned}
$$

Again using (33) and (25) in (51), we get

$$
\begin{aligned}
C &= 4D\left(1-\frac{1}{5}\right) - 16D\left[\frac{1}{(3^5+4.3)} - \frac{1}{(5^5+4.5)} + \frac{1}{(7^5+4.7)} - \cdots\right] \\
&= 16D\left[\frac{1}{(1^5+4.1)} - \frac{1}{(3^5+4.3)} + \frac{1}{(5^5+4.5)} - \frac{1}{(7^5+4.7)} + \cdots\right], \qquad (52)
\end{aligned}
$$

which is the same as the expression (50) given in the above verse. Further, it may be noted that each term in the above transformed Mādhava series involves fifth power of the odd numbers in the denominator, whereas the original Mādhava series had only the first power of odd numbers appearing in the denominator. Thus it goes without saying that the transformed Mādhava series given by (52) would be converging much faster than the original series (16*b*).



4.7 Transformed Mādhava series: II

By choosing $a_p = 2p + 2$ and $a_{p-2} = 2p - 2$ and substituting them in (51), we get a transformed series

$$\begin{aligned} C &= 3D + \frac{4D}{(3^3-3)} - \frac{4D}{(5^3-5)} + \frac{4D}{(7^3-7)} - \cdots \\ &= 4D\left[\frac{3}{4} + \frac{1}{(3^3-3)} - \frac{1}{(5^3-5)} + \frac{1}{(7^3-7)} - \cdots\right], \end{aligned} \tag{53}$$

Figure 7: Graph depicting the convergence of the series given by (52) and (53). Here x-axis represents the number of terms considered in the series.

which again would have faster convergence since the cubes of odd numbers appear in the denominator unlike the first power in the original series. The rate of convergence of the above series in comparison with the series given by (52) is shown in Figure 7. In this figure, as well as in Figures 8 and 9, the series depicted by the graphs are indicated by the beginning of the verse like '*samapañcāhati*' etc., that presents the series.

4.8 Transformed Mādhava series: III

Since the verse "*dvyādiyujāṃ vā*" is composed in terse style, a brief explanation would be in place to facilitate the understanding of its content. Here, the author simply specifies the denominators of the various terms that would constitute the series, without specifying the numerators. This, however, cannot be considered as a lapse or omission on the part of the author, since he simply follows an accepted style of composition in Sanskrit. The numerators can be easily understood to be $4D$, by a process known as *anuṣaṅga* (implicit connection), from the previous verse, that commences with '*vyāsādvāridhinihatāt pṛthagāptam*' which means 'the product of four and diameter separately divided by'.

The denominators of the terms that constitute this series are stated to be squares of the even numbers (starting with two) and diminished by one. Thus, the terms—leaving the first one—constituting the present series may be written as

$$\frac{4D}{(2i)^2 - 1} \qquad (i = 1, 2, \ldots). \tag{54}$$

It is further said that these terms have to be applied to $2D$ positively and negatively, alternatively. Hence, the resulting series would be

$$\begin{aligned} C &= 2D + \frac{4D}{(2^2-1)} - \frac{4D}{(4^2-1)} + \frac{4D}{(6^2-1)} - \cdots \\ &= 4D\left[\frac{1}{2} + \frac{1}{(2^2-1)} - \frac{1}{(4^2-1)} + \frac{1}{(6^2-1)} - \cdots\right] \end{aligned} \tag{55}$$

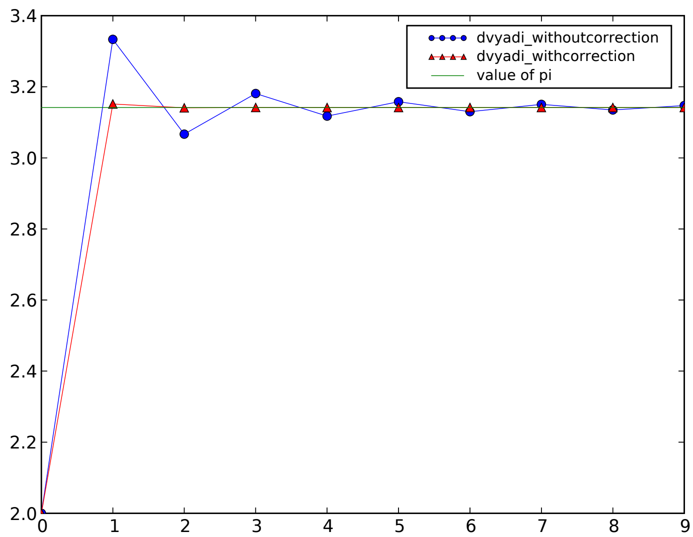


Figure 8: Graph depicting the convergence of the series given by (56) and (57). Here x-axis indicates the number of terms considered in the series.

Here again, the rationale behind the above series can be understood by choosing a_p and a_{p-2} as $2p$ and $2p-4$ respectively and substituting in (51). Doing so, we get

$$\begin{aligned} C &= 4D\left[\frac{1}{2}+\frac{1}{3}-\frac{1}{15}+\frac{1}{35}-\cdots\right] \\ &= 4D\left[\frac{1}{2}+\frac{1}{(2^2-1)}-\frac{1}{(4^2-1)}+\frac{1}{(6^2-1)}-\cdots\right], \end{aligned} \tag{56}$$

which is same as the series given in the verse. Having given the series, in the latter half of the verse it is mentioned that the above series could be terminated by applying a correction-term after dividing by n^2-1, where n is an even number of our choice. The resulting series along with the end-correction would be of the form

$$C \approx 4D\left[\frac{1}{2}+\frac{1}{(2^2-1)}-\ldots+(-1)^{\frac{n-2}{2}}\frac{1}{(n^2-1)}+(-1)^{\frac{n}{2}}\frac{1}{2[(n+1)^2+2]}\right]. \tag{57}$$

It can be easily seen from Figure 8 that the inclusion of the correction-term significantly increases the rate of convergence of the series given by (56).

4.9 Transformed Mādhava series: IV

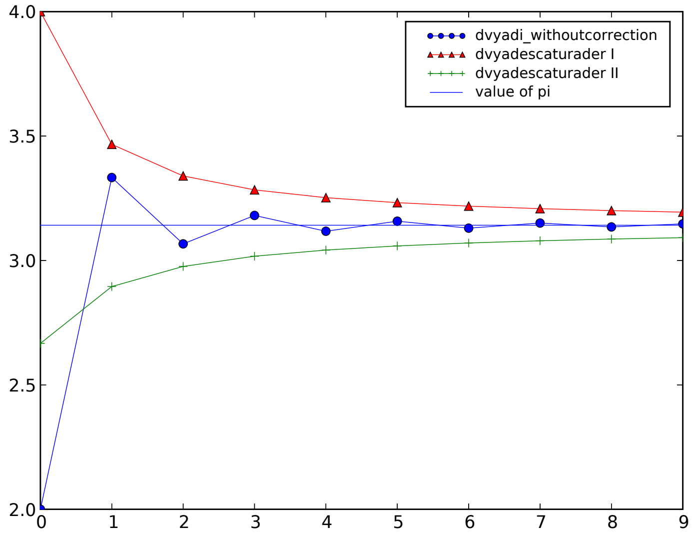


Figure 9: Graph depicting the convergence of the series given by (56) and

its variant forms given by (58) and (59). Here x-axis represents number of terms considered in the series.

The series given by (56) is an alternating series. Mādhava by grouping the terms in the series, seems to have split this into two series, and has represented them as

$$C = 8D\left[\frac{1}{(2^2-1)} + \frac{1}{(6^2-1)} + \frac{1}{(10^2-1)} + \cdots\right]. \quad (58)$$

$$\text{Also,} \quad C = 8D\left[\frac{1}{2} - \frac{1}{(4^2-1)} - \frac{1}{(8^2-1)} - \frac{1}{(12^2-1)} - \cdots\right]. \quad (59)$$

It can be easily seen that the sum of (58) and (59) gives the series (56). Another feature that is noteworthy of (58) and (59) is that, they are monotonically increasing and decreasing respectively (see Figure 9). In fact, both of them asymptotically approach value of the π as shown in the figure.

4.10 Concluding the discussion on Mādhava series

After having a long excursus into the discussion on Mādhava series, in the context of explaining how to find the precise value of the radius (*vyāārdha*) for a given value of the circumference of the circle, the commentator Śaṅkara concludes the discussion by mentioning a rational approximation for the ratio of the circumference to the diameter of a circle. This ratio is specified using the *Kaṭapayādi* system of representing numbers and is given by

$$\frac{C}{D} = \frac{\textit{devalovinayī}}{\textit{mānyastrībāla}} = \frac{104348}{33215} = 3.141592654. \quad (60)$$

It may be noted that the value of π given by (39) is correct to nine decimal places. A discussion on different values of π that have been employed by Indian astronomers may be found in Hayashi's article [14].

Finally, before concluding the paper, we would like to present a glimpse of the interesting discussion on the irrationality of π, found in monumental work *Āryabhaṭīyabhāṣya*—the magnum opus of Nīlakaṇṭha (c. 1500 CE).

5 Irrationality of π

Having specified the ratio of the circumference to the diameter of a circle, Āryabhaṭa in his *Āryabhaṭīya* (c. 499 CE) refers to the value[141] as 'approximate'

[141] The value given is $\frac{62832}{20000} = 3.1416$, correct to four decimal places.

(*āsanna*). Nīlakaṇṭha while commenting upon the verse raises the question: "Why then has an approximate value been mentioned here instead of the actual value?", and then explains [1], p. 41:

> Given a certain unit of measurement in terms of which the diameter (*vyāsa*) specified has no [fractional] part (*niravayava*), the same measure when employed to specify the circumference (*paridhi*) will certainly have a [fractional] part (*sāvayava*). ...
> Even if you go a long way (i.e., keep on reducing the measure of the unit employed), the fractional part [in specifying one of them] will only become very small. A situation in which there will be no [fractional] part is impossible, and this is what is the import [of the expression *āsanna*].

Evidently, what Nīlakaṇṭha is trying to explain here is the incommensurability of the circumference and the diameter of a circle. Particularly, the last line of the above quote—where Nīlakaṇṭha in no uncertain terms mentions that, however small you may choose your unit of measurement to be, the two quantities will never become commensurate—is noteworthy.

6 Conclusion

There has been, and still is, a perception that mathematics in India has just been a handmaiden to astronomy which in turn has been a handmaiden employed in fixing the appropriate times of religious rites. May be true it is; but only partially. If the purpose of mathematics is not broadened to include sheer intellectual excitement, it may be difficult to explain as to why Nīlakaṇṭha cogitated on the irrationality of π or Mādhava evolved elegant methods to obtain the value of π correct to 11 decimal places or much later Śaṅkaravarman (c. 1830 CE), Rājā of Kaḍattanaḍu, specified the value of π correct to 17 decimal places in his *Sadratnamālā*.

It has been well-argued by Raju, in one of his recent publications [10], that the historians of the past have paid little heed to understand and appreciate the distinct approach taken by Indians to mathematics. In fact, on occasions—either due to ignorance or misunderstanding or for reasons not evident to us—the historians have been quite dismissive regarding the Indian contributions as well [3, 9]. But as demonstrated earlier, the contributions of the Kerala school of astronomers and mathematicians to the development of foundations of calculus—in the context of finding the relationship between the circumference and the diameter of a circle—has been quite significant, whose methodology needs to be studied in depth for sheer pedagogical implications, if not for anything else.

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[2] Boyer C B, *History of Mathematics*, John Wiley & Sons, New York 1968.

[3] Boyer C B, *The History of the Calculus and its Conceptual Development*, Dover, New York 1949, pp. 61–62: The Hindus apparently were attracted by the arithmetical and computational aspects of mathematics, rather than by the geometrical and rational features of the subject which had appealed so strongly to the Hellenic mind. Their name for mathematics, *gaṇita*, meaning literally the 'science of calculation' well characterizes this preference. They delighted more in the tricks that could be played with numbers than in the thoughts the mind could produce, so that neither Euclidean geometry nor Aristotelian logic made a strong impression upon them. The Pythagorean problem of the incommensurable, which was of intense interest to Greek geometers, was of little import to Hindu mathematicians, who treated rational and irrational quantities, curvilinear and rectilinear magnitudes indiscriminately.

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[9] Morris Kline, *Mathematical Thought from Ancient to Modern Times*, Oxford 1972, p. 190: (With the Hindus) there is much good procedure and technical facility, but no evidence that they considered proof at all. They had rules, but apparently no logical scruples . . .

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