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MAHĀVĪRA-PHERŪ FORMULA FOR THE SURFACE OF A SPHERE AND SOME OTHER EMPIRICAL RULES

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Several ancient rules for finding the area of geometrical figures are described in the paper. These include the commonly used general-type formula

Area = (perimeter) . (width) / 4 …(1)

for closed round figures and the popular empirical rule

Area = (c+h).h/2

for the circular segment of chord c and height h.

In India, Mahāvīra (9th century AD) gave a practical formula for the concave or convex round surface (e.g. spherical segment) by considering its curvilinear width in (1). It directly leads to the empirical expression $C^2/4$ for the surface of a sphere of greatest circumference C. Interestingly this expression is found in Japan as late as the 17th century AD. Ṭhakkura Pherū of India (about 1300 AD) gave an improved version of the formula.

The process of Jainisation (i.e. transforming a formula to suit the Jaina value $\pi = \sqrt{10}$) has been well illustrated in the paper. Some other rules related to ellipse, sphere, spheroid and ellipsoid (even of higher dimensions) are obtained surprisingly by using the concepts of analogy and averaging.

Key words: Analogical and historical connections; Ancient and medieval mathematics; Practical geometery; Jaina school of mathematics; Spherical and circular segments.

1. Mahāvīra's Formula for Spherical Surface

Since ancient times, the simple formula

$$A_1 = (p.w)/4 \qquad \ldots(1.1)$$

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for the area of a plane round figure has been used. Here p is the perimeter and w is the typical width of the figure. For circular disc or lamina, the width is its diameter d (=2r). Interestingly, the formula (1.1) is also true for a square figure in which case w is taken as the distance between a pair of opposite sides.

More significantly, a practical rule for finding the area A_2 of a segment of a circle (Fig. 1) of arcual length PVQ(=s) and height VM (=h) can be derived

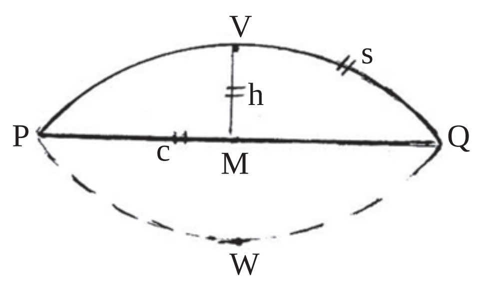

Fig. 1

by using (1.1). The round figure PVQWP (resembling a leaf or ox-eye) is formed by double segment and has p = 2s and w = 2h. So by (1.1) we get the empirical rule

$$2A_2 = (2s.2h)/4$$

or

$$A_2 = (s.h)/2 \qquad \ldots(1.2)$$

Karavinda in his commentary on *Āpastamba Śulbasūtra* has quoted from some ancient Sanskrit work a set of 37 verses one of which has the line[1]

शराहतस्तु कोदण्डो दलितो धनुषः फलम्।

Śarāhatastu kodaṇḍo dalito dhanuṣaḥ phalam.

"The arc multiplied by arrow and then halved gives the area of the bow-figure".

This clearly gives (1.2). But Datta[2] got a different rule by the unusual meaning of *śara* as radius (of the corresponding circular sector). In another quoted line the following popular Indian formula is mentioned to find s in terms of the chord PQ (=c) and h

$$s = \sqrt{c^2 + 6h^2} \qquad \ldots(1.3)$$

Of course, findings of a true relation between c, h, and s was difficult in early days especially in the absence of trigonometry. Based on certain calculations found in the old Babylonian text *BM 85194* (dated about 1600 BC), the present writer (RCG) has shown that the rule

$$s = c + h \qquad \ldots(1.4)$$

was used in antique Babylonia.[3] Considering the double segment (Fig. 1) with p = 2(c+h) in (1.1) or by directly using (1.4) in (1.2) we get the very popular ancient formula

$$A_3 = (c+h).h/2 \qquad \ldots(1.5)$$

for the area of a segment of a circle. This is found in Hellenistic (Egyptian) papyrus (3rd cent. BC), in *Chinese Jiu Zhang Suan Shu* (=*JZSS*) (1st cent. AD), Greek Heron's *Metrica*[4] (where it is attributed to "the ancients"), and in India (see below). Rules (1.2), (1.4), (1.5) are all said to be based on $\pi = 3$ for which they give exact results in the case of a semi-circle.

For quick practical mensuration of an elongated circle (or ellipse of axes 2a and 2b), Mahāvīra (about 850 AD) in his *Gaṇitasāra-saṅgraha* (=*GSS*). VII.21, says[5] "Half the breadth (b) added to the length (2a) and them multiplied by two, is the perimeter (p) of the elongated circle (*āyata vṛtta*). Fourth part of breadth multiplied by perimeter becomes its area."

That is,

$$p = 2\,(2a + b) \qquad \ldots(1.6)$$

and $$A4 = (2b/4).p \qquad \ldots(1.7)$$

Here (1.6) seems to follow the tradition similar to that implied in (1.4) and then the general rule (1.1) is used to get (1.7) with w = 2b. For accurate (*sūkṣma*) calculation, Mahāvīra treats his ellipse like a double circular segment and finds its perimeter by using (1.3) which is based on $\pi = \sqrt{10}$, the popular Jaina value.[6] With this accurate p, the area is found by (1.1).

More significantly, Mahāvīra extended the general rule (1.1) to cover round surfaces which may not be plane e.g. a spherical segment or an (open) umbrela (*Chhatra*) which is one of the eight auspicious (*māṅgalika*) things in every Jaina temple.

For finding the area of the curved surface of a segment of a sphere of radius R (see Fig. 2), the *GSS* VII (*kṣetra*). 25 prescribes the following rule:[7]

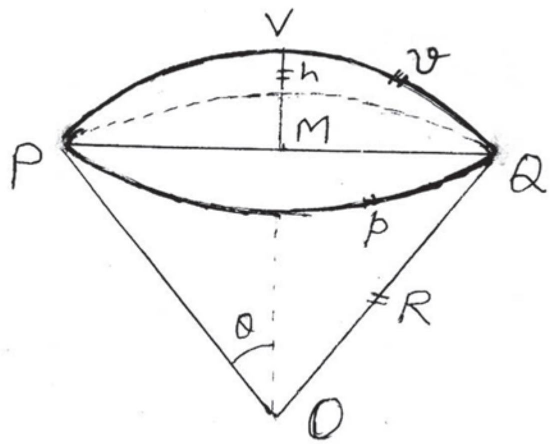

Fig. 2

"Know that one fourth of the perimeter (p of the base circle) multiplied by the curvilinear width (*viṣkambha* or *vyāsa v*) is the area (S_1) of the concave (*nimna*) or convex (*unnata*) surface resembling the fire-pit (*cātvāla*) or the back of a tortoise."

That is,

$$S_1 = (p.v)/4 \qquad \ldots(1.8)$$

In practice, this is a good emprical rule as it takes care for the biggerness of the area of the curved surface more than the diameter of plane circular base would do. The earlier translators of *GSS* such as M. Rangacharya (Madras, 1912) and L.C. Jain (Sholapur, 1963), took *viṣkambha* v as equal to the diameter of the base which is not correct. The two accompanying *GSS* examples ($p_1 = 56$, $v_1 = 27$ and $p_2 = 36$, $v_2 = 15$) show that each pair of p and v is not related to (any) same circle i.e. p/v is not equal to any resonable value of π and etc. Padmavathamma[8] failed to note this and repeated the earlier wrong translation.

It should be noted that as per our interpretation, a given spherical segment will be smaller than, equal to, or bigger than a hemisphere according as v is less than, equal to, or greater than p/2. The case of hemisphere is interesting. If C is the circuference of the base, then p is equal to C and v is C/2. So the area of its curve surface will be, by (1.8),

$$S_2 = C.C/8 \qquad \ldots(1.9)$$

Hence we get the following important empirical formula for the complete surface of a sphere

$$S = C^2/4 \qquad \ldots(1.10)$$

where C is the circumference of any great circle of the sphere. It may be called Mahāvīra's rule. Although not stated explicitly, it follows directly from his (1.8). It appeared in India again in 14^{th} cent. (see below).

Surprisingly, the formula (1.10) appears in Japan as late as in the 17th century.[9] Imamura Chisho gave it in his *Jugai-roku* of 1639, and Isomura in *Ketsugi-sho* of 1660. Significantly, in the second edition (1684) Isomura refers to (1.10) as the old method and further adds that other Japanese mathematicians such as Mori, Yoshida, Imamura Takahara, Hirage, Shimada, Sumida, had not been able to find the correct formula for the surface of a sphere.

Nevertheless it is well known that the famous Greek mathematician Archimedes (died 212 BC) had already derived the exact expression for the surface of a sphere more than 2224 years ago. His result may be concisely put as[10]

$$S_0 = 4\pi R^2 = \pi D^2 = C^2/\pi \qquad \ldots(1.11)$$

In fact he had also found correctly the curved surface of a spherical segment as $2\pi Rh$, where h is the height (VM) of the segment. It may also be mentioned that a rule similar to (1.8) is found also in the Chinese *JZSS* (Chapter I, Problems 33 and 34) but there is disagreement in interpreting it.[11]

2. Some Analogies and Empirical Rules

Analogies or similarities have been playing a significant role in the development of mathematics through the ages.[12] They have often provided methods and techniques for deriving mathematical results whether true or empirical. We have already pointed out that the ancient rule (1.1) applies to circle as well as to square. This very peculiar analogy between the two simple and symmetric plane figures is reflected in another way. Consider a square and its inscribed circle. We have the ratio

R_1 = (area of circle)/(area of square)

$= (\pi d^2/4)/(d^2) = \pi/4$ …(2.1)

and the ratio

R_2 = (perimeter of circle) / (perimeter of sqaure)

$= (\pi d)/(4d) = \pi/4$ …(2.2)

That is, we find that R_1 is exactly equal to R_2! More interestingly, we find a similar analogy in the case of a sphere (of diameter $d = 2r$) and its circumscribed cube (whose side will also be equal to d). Here the ratios are

R_3 = (vol. of sphere) / (vol. of cube)

$= [(\pi/6)\ d^3] / (d^3) = \pi/6$ …(2.3)

and

R_4 = (surface of sphere) / (surface of cube)

$= (\pi d^2)/(6d^2) = \pi/6$ …(2.4)

that is, both these are same again!

It is usually stated that Buddhism was formally introduced in China during the reign of emperor Ming Ti (1st cent. AD) with imperial sanction.[13] There at that time and during the time of Zhang Heng (AD 78-139), the ratio R_1 in (2.1) was taken as 3/4 which implies the use of the simple value π=3 that was common in those days.[14] He was the chief astrologer and minister of the emperor An-ti and seems to have some influence. Following the above ratio, the Chinese, more significantly, surmised that the ratio R_3 in (2.3) must also be the square of 3/4. In other words, the volume of a sphere was taken to be

$V = (9/16)d^3$ …(2.5)

Incidently, this formula explained the rule

$d = (16\ V/9)^{1/3}$

found in the famous Chinese classic *JZSS*. Rule (2.5) implies $R_3 = 9/16 = 0.5625$ while the correct value is $\pi/6 = 0.5236$ nearly.

It is well known that in India, Āryabhaṭa I (born 476 AD) in his *Āryabhaṭīya* (II.7) gave his peculiar rule for the volume of a sphere as[15]

$$V_1 = A\sqrt{A} = (\pi r^2)\ \sqrt{\pi r^2} \qquad \ldots(2.6)$$

where A is the area of a central section. This is an empirical rule although he called it exact (*niravaśeṣa*). His commentator Bhāskara I (629 AD) quotes the Sanskrit text of a practical (*vyāvahārika*) rule which can be expressed as

$$V_2 = (9/2).(d/2)^3 \qquad \ldots(2.7)$$

This is same as (2.5). He does not specify the Indian source for the quotation and seems to be of the opinion that Āryabhaṭa attempted to find his rule after noting the empirical nature of the then known rule(s).

Since *JZSS* invariably uses π=3, the rule (2.5) was taken, as suggested by its commentator Liu Hui (263 AD), equivalently as[16]

$$V_3 = (\pi/4)^2.d^3 = (\pi^2/16).d^3 \qquad \ldots(2.8)$$

Although (2.5) gives V in excess, the calendarist Zhang thought it otherwise and further made it worse by taking[17]

$$V_4 = (9/16)d^3 + (1/16).d^3 \qquad \ldots(2.9)$$

$$= (5/8)/d^3 \qquad \ldots(2.10)$$

Comparing this with (2.8) leads to the value $\pi = 10$ which is not only better than $\pi = 3$, but even better than Zhang's another value $\pi = 92/29$.

There is another consideration. Suppose the sphere is inscribed in a right circular cylinder whose height (and diameter of the base also) is d. Moreover, let this cylinder be circumscribed by a cube of side d such that their bases and tops coincide. It was known (and which is also clear from the ratio R_1) that:

(volume of cylinder) / (volume of cube)

$$= [\pi d^2/4).d]/(d^3) = \pi/4 \qquad \ldots(2.11)$$

Now let

$$(\text{volume of sphere}) / (\text{volume of cube}) = k \qquad \ldots(2.12)$$

From these two equations we get the volume of sphere

$$V_5 = (\pi k/4).d^3 \qquad \ldots(2.13)$$

More than two millennium ago, Archimedes[18] had already derived the exact value $k = 2/3$.

In fact, he considered this mathematical discovery to be his greatest achievement and so this ratio along with the figure of sphere and cylinder was (according to his will) inscribed on his tomb. Zhang took the value of k as $\pi/4$ or 3/4. Liu Hui was aware that this assumption is wrong although he could not find the correct value.

There are surprising cases of analogies of the ratios R_1 to R_4. Let an ellipse (of axes 2a and 2b) be inscribed in a rectangle whose sides are parallel to the axes. Here the ratio R_1 is

$$\text{(area of ellipse)} / \text{(area of rectangle)} = \pi ab/4ab = \pi/4$$

which is same as in the case of a circle!

If q is the perimeter of the ellipse, we have, on presuming analogy similar to R_2 in (2.2),

$$q/\text{(perimeter of rectangle)} = \pi/4$$

This yields

$$q = \pi\ (a+b) \qquad \ldots(2.14)$$

which is not correct and so the analogy breaks down in this case. However, the empirical rule (2.14) is same as was obtained by Kepler (about 1600 AD) by taking the mean of the axes because he could not do better.[19] Here the process of averaging is supported by analogy just as it is theoretically supported by method of least squares.

In the case of ellipsoid of axes 2a, 2b, and 2c, inscribed in a cuboid (of same sides) we have

$$\text{(volume of ellipsoid)} / \text{(volume of cuboid)}$$

$$= [(4\pi/3)\ abc] / (8\ abc) = \pi/6 \qquad \ldots(2.15)$$

which happens to be same as R_3 in (2.3)! So here also we can use the analogy represented by R_4 in (2.4) to find empirically the curved surface S of the ellipsoid. We will have

$$S / \text{(surface of cuboid)} = \pi/6 \qquad \ldots(2.16)$$

The total surface of the cuboid will be the sum of the six faces (of the rectangular solid) and will be (8ab + 8bc + 8ca). Hence (2.16) gives

$$S = 4\pi.(ab+bc+ca)/3 \qquad \ldots(2.17)$$

which again implies a sort of averaging. For an spheroid or ellipsoid of revolution, we take b = c in (2.17) thereby getting.

$$S = 4\pi\ (2ab+b^2)/3 \qquad \ldots(2.18)$$

which is found in Japan quite late.[20]

3. The Process of Jainisation

In the history of mathematics, empirical rules have been often modified in the name of improvement and to suit some specific situation or purpose. Consider the familiar case of the circular segment PVQ (Fig. 3). The enclosed area of the

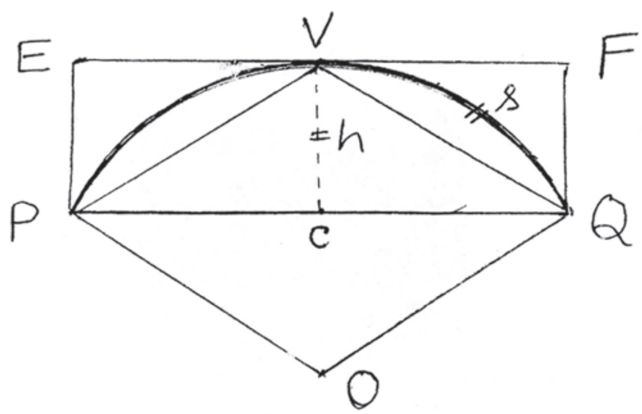

Fig. 3

curved segment PVQP clearly lies between the area of the inscribed ΔPVQ (=ch/2) and that of the circumscribed rectangle PEFQ (=c.h). By the usual averaging, we get the rule

$$A = (3/4)\ ch \qquad \ldots(3.1)$$

empirically for the area of the segment.

Incidently we find that (3.1) gives correct area for the semicircle (c = 2r, h = r) with the commonly used simple value $\pi = 3$.

In the Jaina school, $\pi = \sqrt{10}$ is considered accurate. So (3.1) may be modified by multiplying it by $(\sqrt{10}/3)$ to get the desired rule

$$A_1 = \sqrt{10}.(ch/h) \quad \ldots(3.2)$$

Rules equivalent to (3.2) are found[21] in the *Tiloyapaṇṇatti* (IV. 2401) of Yativṛṣabha (placed between 443 and 609 AD) and in *Bṛhatkṣetra-samāsa* (I.122) of Jinabhadra Gaṇi (about 609 AD). The first half of a Prakrit *gāthā* quoted by Bhāskara I in his commentary (629 AD) on the *Āryabhaṭīya* (under II.10) also gives (3.2) in this form.[22] Mahāvīra mentions it as an accurate rule in his *GSS* VII.70 while (1.5) of Section 1 is considered approximate (*GSS*. VII.43). The same is the case with Nemicandra's *Trilokasāra* (*gāthā* 762).[23]

Take another example. Similar to the linear Babylonian rule (1.4), there seems to be another simple empirical rule apparently obtained by analogy of segment and semicircle. It is

$$s_1 = 3\ [c/4) + (h/2)] \quad \ldots(3.3)$$

Being based on $\pi = 3$, its Jaina form will be

$$s_2 = \sqrt{10.}[(c/4) + (h/2)] \quad \ldots(3.4)$$

The Sanskrit line for this ancient rule is quoted by Bhāskara I in his above mentioned commentary.[24]

The process of modifying a formula to adjust it to the Jaina value $\pi = \sqrt{10}$ is called Jainisation for convenience. Of course, an adjustment can be done for other values of π also. To illustrate this historically, we take the case of the popular ancient rule (1.5) which is based on π=3 as already detailed in Section 1 above. It may be modified to any general value of π by using the multiplying factor $f = \pi/3$. The resulting empirical rule is

$$A_0 = (c+h).(h/2).(\pi/3) \quad \ldots(3.5)$$

The Jainised form of this will be

$$A = (c+h).(h/2).(\sqrt{10}/3) \quad \ldots(3.6)$$

which was quite popular in India. Its equivalent mathematical form is found in[25]

(i) *Triśatika*, *sūtra* 47 of Śrīdhara (about 750 AD);

(ii) *Gaṇitasāra*, III.46 of Ṭhakkura Pherū (about 1300);

(iii) *Mahāsiddhānta*, XV.89 of Āryabhaṭa II whose date has been now shifted to about 1500 AD;

(iv) The anonymous *Pañcaviṃśatika* (A_{24}).

In the last work another Jainised form based on $\pi = 19/6$ is mentioned.[26] This value of π is usually found by using the rule

$$\sqrt{a^2+x} = a + (x/2a) \qquad \ldots(3.7)$$

with a = 3 and x = 1 here. A form of (3.5) based on $\pi = 22/7$ is found in Mahāsiddhānta, XV.93 of Āryabhaṭa II while that based on $\pi = 63/20$ was given by Keśava and etc.[27] This value of π (i.e. 3.15) is roughly mean of 22/7 = 3.14 nearly and $\sqrt{10} = 3.16$ nearly.

In India, both Mahāvīra and Nemicandra did not consider the formula (1.5) worthy for accurate calculations. In China, great regard was given to it because it was found in *JZSS* which was considered a sort of mathematical bible there. Its use has been suggested in deriving another rule.

From Fig. 3 we see that in areas

sector OPVQ = ΔOPQ + segment PVQ

or, $\frac{1}{2}sr = \frac{1}{2}\, c\,(r-h) + \frac{1}{2}\,(c+h)\, h$

which on simplification yields

$$s = c + h^2/r = c + 2h^2/d \qquad \ldots(3.8)$$

This formula is found in the *Mengqi Bitan* ("Dream Pool Essays") of Shen Gua (11th cent. AD) of China.

Now consider the following rectification rule

$$s_0 = \sqrt{c^2 + kh^2} \qquad \ldots(3.9)$$

For k = 4, it gives the length of the broken Chord PVQ. For the same k, it will give, by (3.7)

$$s_3 = c + 2h^2/c \qquad \ldots(3.10)$$

which is comparable with Shen Gua's (3.8). For semicircle c = d = 2r, and h = r; and both the rules yield 3r which implies $\pi = 3$. Interestingly, in an Italian anomymous manuscript of the 15th century, (3.10) was adjusted to π as

$$s_4 = (\pi/3).(c+2h^2/c) \qquad \ldots(3.11)$$

In India on the other hand, the basic rule (3.9) itself was adjusted to various values including the Jaina $\pi = \sqrt{10}$ which is implied in (1.3). The details of $k = (\pi^2-4)$ etc. have been published.[29]

Formula (3.6) represents historical Jainisation of a circular segment rule. A case of spherical surface may be taken. By equating the left hand sides of the analogy rules (2.3) and (2.4), we get

Surface of sphere = $(6d^2/d^3)$.(vol. of sphere)

Using the next equation (2.5), this surface is

$$S = (6/d).(9/16).d^3 = (27/8)d^2 \qquad \ldots(3.12)$$

For Jaina value, circumference $C = \sqrt{10}d$, so that the above becomes

$$S = (27/80).C^2 \qquad \ldots(3.13)$$

This Jainised formula is closer to the true (1.11) than Mahāvīra's (1.10) is!

Thus we see that Jainisation can be done and affected in different ways. Of course, the usual way of Jainisation of a formula based on $\pi = 3$ is to use the factor $\sqrt{10}/3$ or 19/18 (linearly). However, if the formula to be Jainised involves the square of π (or of C), the multiplying factor was taken to be 10/9 (which is square of $\sqrt{10}/3$) etc. Nevertheless, one has to be careful about mathematical coherency along with some historical facts.

For the volume of a sphere of diameter d (= 2r), the empirical formula

$$V = (9/16).d^3 = (9/2).r^3 \qquad \ldots(3.14)$$

which is verbally quoted by Bhāskara I (629 AD), was quite popular in the Jaina School and is already mentioned above in Section 2. Mahāvīra in his *GSS* VIII.28, has given it as a practical (*vyavahārika*) formula not meant for accurate (*suksma*) calculations. Later on the same formula appears in the *Tīoyasāra* (*gāthā* 19) of the Jaina Nemicandra (10th cent. AD). Its occurrence in China is already discussed above (see Section 2).

That the formula (3.14) was considered to involve π^2 is amply illustrated by the present writer (RCG) in his paper published in the *Journal of the Asiatic Society*.[30] Mahāvīra also seems to fall a prey to the ancient thinking of taking (3.14) to involve the square of π. So he tried to improve it by the Jainisation factor 10/9 thereby expecting to make it accurate (i.e. based on p=√10, instead of π=3).

The orginal Sanskrit text (as found in the manuscripts) of his improved rule runs as[31]

तन्नवमांशं दशगुणमशेषसूक्ष्मं फलं भवति।

Tannavamāṃśaṃ daśguṇamaśeṣasūksmaṃ phalaṃ bhavati.

"The ninth part of that (rough value V just found in the previous line) multiplied by ten becomes the accurate volume without remainder."

that is,

accurate volume $V_1 = (V/9).10$ …(3.15)

$= 5r^3$ …(3.16)

Interestingly, the expression (3.16) can also be arrived at by using Āryabhaṭa's (2.6) as follows. Putting the common simple $\pi = 3$, we get

$$V_2 = 3r^2 .\sqrt{3r^2} = 3\sqrt{3r^3} \quad \dots(3.17)$$

Now applying the approximation

$$\sqrt{a^2+x} = a + x/(2a+1) \quad \dots(3.18)$$

with a = 1, x = 2, we get $\sqrt{3} = 5/3$ which, when put in (3.17), leads us to the desired expression (3.16). Like the Chinese Zhang's effort represented by the equation (2.9), Mahāvīra's attempt did not improve (3.14). In fact, we get from bad to a worse result because both had a false impression and estimation of (3.14).

4. Pherū's Formula's for Sphere

The exact mathematical formula

$$V_0 = (\pi/6).d^3 \quad \dots(4.1)$$

for the volume of a sphere of diameter d (=2r) was known to the Greek Archimedes (died 212 BC). It was also known in China in the early sixth century AD[32] For the simple value $\pi = 3$, it will reduce to the equally simple form

$$V_1 = (d^3)/2 \quad \dots(4.2)$$

A pragmatic interpretation of this is that the volume of a sphere is practically just half of the volume of a cube of side d. The approximate truth can be easily verified by even a crude weighing of their simple models made of the same material (say clay). Theoretically, (4.2) could be arrived by derivation based on some starting result in which $\pi = 3$ is implied as was the case in China.[33] In any case, the rule (4.2) may be taken to be the simple ancient formula for the volume of a sphere just as 3d was taken as the circumference of a circle.

The usual Jainisation of (4.2) will lead us to the following two formulas

$$V_2 = (d^3/2).(\sqrt{10/3}) \quad \ldots(4.3)$$

$$V_3 = (d^3/2).(19/18) \quad \ldots(4.4)$$

Of course, the second can be obtained from the first one by applying the frequently used rule (3.7) of Jaina mathematics.

The formula (4.4) is quite convenient. It is found in the *Triśatikā* (*sūtra* 56) of Śrīdhara, in the *Siddhanta-Śekhara* (13.46) of Śrīpati (11th centruy AD), and etc.[34] It may be pointed out that Śrīdhara's Jainised formula (3.6) was obtained by using the factor √10/3 while his (4.4) implies 19/18. Of Course, it may be that he knew the exact mathematical formula (4.1) from which he got (4.4) either directly or through $\pi = \sqrt{10}$. But then the question would arise about his source or proof of (4.1).

The case of Ṭhakkura Pherū (early 14th century AD) is more interesting. The earlier traditional formula (3.14) is found in his *Gaṇitasāra* (5.25a) in a slightly different form.[35]

$$V_4 = (3/4)\ (3/4)d^3 \quad \ldots(4.5)$$

But he seems to be fully aware that this gives the volume of a sphere in excess and aslo that the use of Mahāvīra's Jainisation factor 10/9 would make it worse. So he did not give Mahāvīra's formula (3.15). He tried to improve the rough rule (4.2) which gives the volume in defect. He might have picked up this expression from Śrīdhara's Jainised form (4.4) or may have found it by weighing etc.

Unfortunately, in stead of using some proper factor (say like √10/3 or 19/18 used earlier by Śrīdhara etc.), he still revered Mahāvīra's factor 10/9. So he gave the following formula in his *Gaṇitasāra*[36]

$$V_5 = (d^3/2).(10/9) \quad \ldots(4.6)$$

Thus Pherū could change the defective formula (4.2) to one in excess but could not obtain greater accuracy like those of (4.3) and (4.4). However, his (4.6) is slightly better than the traditional rule (4.5) which he also included in his work. Interestingly he seems to be aware of the merit of (4.6) over (4.5). So the numerical example with d= 6, although given just after (4.5), was still solved by his newly Jainised rule (4.6) which yields the given answer of 120. But had he used the linear factor $\sqrt{10}/3$ (instead of the square form 10/9) in Jainising the rough rule (4.2), he would have got a result better than even Śrīdhara's.

It seems that Pherū had great faith in the Jainisation factor 10/9. So he applied it to improve even the rule (1.10) for the surface of a sphere (see Section 1 above). The new formula will be (in modern notation)

$$S = (C^2/4).(10/9) \qquad \ldots(4.7)$$

$$= C^2/(3.6) \qquad \ldots(4.8)$$

The formula (4.7) is found in *Gaṇitasāra*[37] at two places (3.65 and 5.25) and is an improvement on Mahāvīra's (1.10). It may be called Mahāvīra-Pherū Formula for the surface area of a sphere.

5. Epilogue

The exact mathematical formula for the volume of a sphere and the curved surface of a spherical segment were already proved in Greece by Archimedes more than 2224 years ago. Yet quite a few various type of empirical rules in this regard are found in other cultures later on. This shows that either others were not aware of the grand Greek achievements in the field or they preferred to follow different traditions and lines of thoughts. Occurrence of a specific or typical formula in various culture areas implies many possibilities such as independent discovery or invention, borrowing and transmission, or drawing from a common source etc. Regarding the popular empirical rule $V = (9/12)\ r^3$ for the volume of a sphere, Jadhav[38] writes that very likely "it might have been transmitted from China to India where it might have been held in the Jaina School of Mathematics." And although there were contacts between the two countries, more positive evidence (other than circumstantial and chronological priority) may be needed to draw further conclusions.

In India, Bhāskara II (12th century AD) has not only given the exact rules

$$S = 4\pi r^2, \text{ and } V = S.d/6 \qquad \ldots(5.1)$$

in his *Līlāvatī* (*sūtra* 201) but has also given their derivations elsewhere.[39] Based on the popular Archimedean approximatism $\pi = 22/7$, the expression $11d^3/21$ was also used in the various parts of the world during medieval times (even Śrīdhara is credited for knowing $\pi = 22/7$).[40] Jainas (e.g. Ṭhakkura Pherū) continued to give empirical rules for mensuration of sphere even after Bhāskara II in their traditional type of works where some novelties were also added. In our therminology Zhang of China (about 100 AD) was just doing Jainisation when he changed the expression $(9/16)d^3$ (for the volume of a sphere) to $(5/8)d^3$ by increasing the former by its own one-ninth (see Section II for details). He in fact knew the so called Jaina value $\pi = \sqrt{10}$. Jadhav[41] has given details of occurrence of this value in various Jaina works with their dates such as *Sūrya-prajñapti* (about 500 BC), *Jīvabhigama-sūtra* (500 BC?), *Bhagavatī-sūtra* (300 BC?), etc. However, it must be noted that Jaina Āgamas (canonical literature) although composed and compiled quite early (about 300 BC according to Hermann Jacobi), were continued to be modified and edited upto the 5th century AD.[42] The problem of early Indian chronology has been always a chronic problem.

The relation between the volume V and surface S of a sphere of radius r can be concisely expressed in modern calculus notation as differentiation namely

$$dv/dr = S \qquad \ldots(5.2)$$

using which one can be found from the other easily. The important point to note is that (5.2) holds good for hyperspheres also i.e. for spheres in n dimensions(= n - D briefly). For example, the V and S of a 4-D sphere are known to be[43]

$$V = (\pi^2/2).r^4 \qquad \ldots(5.3)$$

$$S = (2\pi^2).r^3 \qquad \ldots(5.4)$$

For a n - D sphere, its volume will be of the form $k\,r^n$. Whether we are able to imagine some physical and geometrical meanings of these hyperspheres or not, is a separate matter. But theoretically and mathematically there seems to be not much difficulty in extending many concepts of geometry (or of vectors etc.) from there to more dimensions. e.g. the equation of a 4 - D sphere in 4 - D coordinate geometry is taken as

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = r^2 \qquad \ldots(5.5)$$

Details are to be considered and worked out carefully keeping in view the norms of metrology or of units and dimensions of ‘measurements’ involved. A

4-D 'cube' of side d(= 2r) will have 'volume'd^4 whose dimension will be L^4 (as dimension of d is L). The number of 'faces' of a 4 - D cube will be eight (a n - d cube will have 2n faces). the area of each face will be d^3 and its total surface $8d^3$. In this way the nature's analogies as mentioned in Section 2 can be worked out and checked. For instance in the case of 4 - D, the ratio,

(vol. of sphere)/(vol. of cube)

$$= [(\pi^2/2).r^4]/d^4 = \pi^2/32 \qquad \ldots(5.6)$$

Also the ratio for their surfaces,

(Surface of sphere) / (total surface of cube)

$$= (2\pi^2 r^3) / (8d^3) = \pi^2/32 \qquad \ldots(5.7)$$

which is exactly same as (5.6)! In fact, the analogy[44] holds good for all hyperspheres. As illustrated in Section 2 of the paper, we can use this analogy to find rules (whether true or empirical) for the volumes and surfaces of other round solids including hyper-ellipsoids. Historical studies and perspectives do help in motivating mathematicians for finding more things in mathematics and in forming fresh problems for investigations (e.g. number of edges and corners of n - D cube).

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