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TREND OF GEOMETRICAL RESEARCHES IN CALCUTTA UNIVERSITY: 1881–1931

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The contribution of Sir Asutosh Mookerjee in Calculus and Geometry and of his younger colleagues like Prof. Symadas Mukhopadhyay in the fields of Synthetic, Hyperbolic and Differential Geometry, Prof. R.C. Bose, in Hyperbolic and Non-Euclidean Geometry, Prof. R.N. Sen in Deductive Geometry have been emphasized to have an idea of Geometrical researches in the period 1881-1931 in the Calcutta University.

Sir Asutosh Mookerjee, India's premier mathematical researcher had first initiated research in Geometry at a time, when research in any area of mathematics was practically an unknown entity in India. His mathematical investigations formally began in 1881. With no formal guide to help him, Asutosh Mookerjee's gestation period of mathematical creativity can be discovered from his "Diary". He identified some great celebrities of mathematics, like Joseph Luis Lagrange (1736-1813), Andre – Marie Legendre (1752-1833), Pierce Simon Laplace (1749-1827), Gaspard Monge (1746-1818), Carl Frederick Gauss (1777-1855) etc, who incidentally formed the upper crust of the European mathematical community. He read their memoirs and treatises and his fascination for geometrical studies was inclined more towards French mathematics. The seminal influence of G. Monge on Asutosh Mookerjee was perhaps most responsible for the later's monumental contributions in the field of Geometry. It may be noted that A.M. Legendre was the author of a masterly text on "*Elements de Geometries*", which dealt with the essentials of Euclidean Geometry. Influenced by such work, Asutosh too gave an elegant new proof of the 25th proposition of the first book of Euclid. It came out in the "*Messenger of Mathematics*" 10(1881), 122-123. Asutosh was then a mere boy of seventeen. In 1883–84, he published his second

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paper named "Extension of the theorem of Salmon" again in the "*Messenger of Mathematics*", 13, 157-160 .

In this second paper, he gave some extension of a theorem of Salmon. Being enamored by the works of Lagrange, Asutosh identified elliptic functions as one of his areas of research. Influenced by G. Monge, Asutosh began doing research by coupling geometry and calculus and this resulted in several very important publications. They may be listed as follows:

- "A note on elliptic functions" [*QJPAM*, 21(1886), 212-217]. In this paper, he proved some addition theorem in the theory of elliptic functions by a new elegant method on the properties of the ellipse. Enneper in his book "*Elliptische Funktionen*" has referred to this paper by Asutosh.
- "On the differential equation of a trajectory" [*JASB*, 56(1887), 117-120]. In this paper, Sir Asutosh gave simple method of solution of the differential equation of oblique trajectory of confocal ellipses. This was a far better method as compared to the one given earlier by the Italian Mathematician Mainardi. A.R. Forsyth in his book *Differential Equations*, has referred to Sir Mookerjee's treatment.

A born geometer, Asutosh Mookerjee went on systematizing the applications of differential equations to different types of relevant conics. The following papers are noteworthy in this connection

- "On Monge's differential equations to all conics" [*JASB*, 56(1887) 117–120].
- "A memoir on plane analytical geometry" [*JASB*, 56(1887) 288–349].
- "A general theorem on the differential equations of trajectories" [*JASB*, 57(1888) 72–99].
- "Remarks on Monge's differential equation to all conics" [*PASB*, February, 1888].
- "The geometric interpretation of Monge's differential equations to all conics" [*JASB*, 58(1889) 181-185].
- "On the differential equation of all parabolas" [*JASB*, 57(1888) 316-332].
- "In a curve of aberrancy" [*JASB*, 59(1890) 61-63].

It is noteworthy, that without any guidance from anywhere, depending exclusively on his own brilliant mind, Asutosh successfully gave geometrical interpretation of Monge's Differential Equations to all conics. He interpreted that "the radius of curvature of the aberrancy curve vanishes at every point of every conic."

This unique contribution of Sir Asutosh has been referred to in Edward's textbook on Calculus. In 1893 Sir Asutosh published a textbook on Geometrical Conics. The book was quite popular in those days amongst the students.

It may also be noted that Asutosh Mookerjee's research papers roused a lot of interest among a section of mathematicians. In a letter written to Sir Asutosh from Cambridge, dated the 14th September, 1887, A. Cayley in a way supported Sir Asutosh's criticism of J. J. Sylvester's interpretation of the Mongian and wrote " it is of course all perfectly right'. Col. Cunnigham wrote "Professor Asutosh Mukhopadhyay has proposed a really excellent mode of geometric interpretation of differential equation in general……."

Asutosh Mookerjee was a trendsetter in the field of geometrical research in Calcutta. He inculcated the spirit of enquiry in the younger mathematicians. Syamadas Mukhopadhyay (1866-1937) a brilliant young mathematician was brought in by Sir Asutosh Mookerjee and he joined the newly formed Department of Pure Mathematics of the University of Calcutta as a lecturer in 1911. Sir Asutosh was apparently his idol. Greatly influenced and encouraged by Sir Mookerjee, Syamadas took up research in various branches of Geometry and his contributions were remarkable. In fact in one of his papers entitled "On rates of variation of the osculating conic" [*BCMS*, 1.2(1909) 125–130], Syamadas Mukhopadhyay has referred to a paper by Sir Asutosh Mookerjee entitled "The Geometric interpretation of Monge's differential equation to all conics" [*JASB*,58.2,(1889) 181–185] and has commented that "Prof. A. Mukhopadhyay has based an elegant interpretation of the differential equation of the general conic".

Professor Syamadas Mukhopadhyay made outstanding contributions in the field of Geometry. As remarked by another great Pure Mathematician of Calcutta University Prof. M.C. Chaki (1913-2007), "His contributions in geometry stand out prominently on account of their novelty and originality".

Syamadas Mukhopadhyay's research work may be broadly classified into three parts. In the first part, we place his researches dealing synthetically with properties of a plane curve, specially in infinitesimal regions. Here he developed new methods. These methods led to a number of interesting theorems on the existence of cyclic and sextactic points on a convex oval. The theorem which states that the minimum number of cyclic points on a convex oval is four and sextactic point is six, is now well known and is referred to in geometry as Mukhopadhyay's "Four Vertex Theorem". The proof of this celebrated theorem was published in the *BCMS* as "New methods in the geometry of a plane arc, I. cyclic and sextactic points" [*BCMS*, 1(1909), 31–37].

This paper & the subsequent three papers:

- "On rates of variation of the osculating conic" [*BCMS*, 1(1909) 125–130].
- "Parametric coefficients in the differential geometry of curves in an N – space, I, general conceptions" [*BCMS*, 2(1909) 187–200].
- "Parametric coefficients in the differential geometry of curves in an N – space, II, extension of Serret Frenet Formulae to curves in an N – dimensional space" [*BCMS*, 1.4(1909),233-243] dealing with these methods of analytical geometry were all published in the *BCMS* Vol. 1, 1909. These new methods developed by Prof. Mukhopadhyay aroused a good deal of interest among some famous mathematicians of the time. Prof. Hadamard referred to his work in the memoirs of College de France. Prof. W. Blaschke gave credit to Prof. Mukhopadhyay for giving the first proof of the celebrated "Four Vertex Theorem". Prof. Engel also showed much interest in these developments.

Later on Dr. Mukhopadhyay generalized these results. He proved that if a conic (circle) meets an oval in 2n points then the oval has 2w cyclic (sextactic) points. The connection between cyclic points and normal were also studied by him. His papers entitled:

- "New methods in the geometry of a plane arc II, cyclic points and normals" [*BCMS*, 10(1919) 65-72, 1919].
- "Extended minimum number theorems of cyclic and sextactic on a plane convex oval" [*Mathematische Zeitschrift*, Band 33(1931) 648–62].
- "Circles incident on an oval of undefined curvature" [*Tohuku Journal of Mathematics*, Vol. 37, 1931].

are all noteworthy publications in this context.

Professor R.C. Bose (1901-1987), a famous student of the Department of Pure Mathematics, first started working under Professor S. Mukhopadhyay & he subsequently proved that sextactic points cannot all lie on a conic. The properties of convex ovals and ovaloids in relation to the various kinds of centroids have been studied by R. C. Bose and S. N. Roy. They gave special emphasis to the loci of these centroids for a system of parallel ovals or ovaloids.

The second part of the research work of Mukhopadhyay belongs to the field of plane hyperbolic geometry. Mukhopadhyay's paper entitled

- "Geometrical investigations on the correspondence between a right – angled triangle a three right – angled quadrilateral and a rectangular pentagon in Hyperbolic Geometry" [*BCMS*, XIII. 4(1922-23) 211-216] deserves special mention.

As is known from Gaussian pentagram of elliptic geometry, there exist five right – angled triangles and five tri – rectangular quadrilaterals of hyperbolic plane. S. Mukhopadhyay in the above mentioned paper completed this system by showing the existence of a five – right – angled hyperbolic plane pentagon associated to the other eleven figures and thus gave an objective basis for Engel – Napier rules. In his own words, at the end of the paper, he concludes "We have thus the closed series of five associated right – angled triangles and the Engel – Napier Rules are shown to possess a real geometrical basis in the rectangular pentagon".

In 1926, with the publication of the paper entitled "On general theorems of co–intimacy of symmetries of a hyperbolic triad" [*BCMS*, 18(1926) 39-55] started a very fruitful collaboration between Syamadas Mukhopadhyay and his legendary student R.C. Bose.

They discussed in that paper the scope to extend to all hyperbolic triads the well known Concurrency theorems of the angle bisectors and the right bisectors of the sides a triangle formed by three line elements meeting at three actual vertices.

Actually, the extension of the angle bisector theorem to all possible triads of linear elements meeting at actual, improper or ideal vertices was effected by using pure geometry in an earlier paper jointly written by Prof. Syamadas Mukhopadhyay and his student Gurudas Bhar . The paper entitled "Generalization of certain theorems in the hyperbolic geometry of the triangle "[*BCMS* 12.1(1920–21) 14-28]. Under the guidance and encouragement of Prof. Mukhopadhyay,

Gurudas Bhar carried out more researches in geometry. Two more papers published by G. Bhar, namely

- "Geometrical construction for the limiting centres of a cubic" [*BCMS*, 12 (1920–21) 1-4]
- "The osculating conic in homogeneous co-ordinates" [*BCMS*, 12 (1920–21) 85-104]

bear testimony to that.

R.C. Bose published the following papers:

- "New methods in euclidean geometry of four dimensions" [*BCMS*, 17 (1926) 105-140]
- "The theory of associated figures in hyperbolic geometry" [*BCMS*, 19 (1928) 101-116]
- "The theory of skewed regular pentagons in hyperbolic space. Combinations of associated pentagons" [*BCMS*, 28 (1936) 159-186]

He went on to make many important contributions in the field of non – Euclidean Geometry and established himself as a Geometer of repute. Prof. Syamadas Mukhopadhyay's contribution in nurturing R.C. Bose and encouraging him to carry forward the tradition of geometrical research in the University of Calcutta deserves special mention. Later on R.C. Bose was picked up by the pioneer statistician P.C. Mahalanobis for his sound knowledge of Pure Mathematics in general and geometry in particular and R.C.Bose became a renowned statistician in the years to come. But these discussions are not needed here.

Now we again go back to Prof. Mukhopadhyay's contributions to make Calcutta University an important centre for research in Geometry. The third and final part of his research work deals with differential geometry of curves. For space curves of n dimensions Prof. Mukhopadhyay introduced certain differential forms. He called them "Parametric Coefficients" and using these he expressed many invariant property of curves.

He suggested a stereoscopic device for visualising in 4 – dimensional space. He wrote a paper entitled "A note on the stereoscopic representation of four dimensional space" [*BCMS*, 4 (1912-13) 15]. Professor Bryan criticized the work and Prof. Mukhopadhyay gave a scientific reply published as "Reply to Professor Bryan's criticism" [*BCMS*, 6 (1914-15) 55–56].

In 1931–32, Professor S. Mukhopadhyay published two important papers entitled "Lower segments of M curves" and "Cyclic curves of an ellipsoid" [*JIMS*, 19 (1931) 75–80]. In the first of these research papers, Prof. Mukhopadhyay explains that the name M-curve (Monotropic Curve) is due to Stackel. H. Mohrmann, who in adopting the name, gives the following precise definition.

> "A singularity – free (that which does not intersect itself) real branch of an analytic curve, which divides the Euclidean plane into two and only two regions, and for which the curvature at every finite point is limited and different from zero, will be called a limited monotropic curve or simply an M-curve". (*Mathematische Annalen*)

In his investigation, Syamadas Mukhopadhyay considered M – curves which are not necessarily analytic. He has characterized the M – curves in a different way and investigated the associated problems.

As regards the second publication viz "Cyclic curves on an ellipsoid", [*JIMS*, 19 (1932) 246–250], Prof. W. Blaschke suggested in his lecture on "Selected problems of differential geometry" (1932), that the object of all Mukhopadhyay's paper is to study certain properties of cyclic curves on an ellipsoid, which is known to possess six vertices.

These discussions clearly reveal the kind of excitement and interest that was generated by the research publications of Syamadas Mukhopadhyay in the mathematical world of the twentieth century.

On one hand his personal achievements added to the glory of the newly set up department of Pure Mathematics of the University of Calcutta, while on the other hand, his guidance, encouragement and suggestions brought forth a very renowned set of research workers on different branches of geometry and made Calcutta University an important centre of geometrical research for a long time to come.

Another pioneer mathematician **Dr. Haridas Bagchi** (1888-1968) also joined the newly formed Department of Pure Mathematics as a lecturer in 1911. Haridas Bagchi was also drawn towards Geometry. In 1912, he was the second scholar to obtain Ph.D degree from the Calcutta University in Mathematics and the title of his thesis was "Curves of the third order, cubics and quartics".In the earlier part of his career, Dr. Bagchi carried out research on cubics, cycloid and hypercycloids.

Just to give an idea, about the type of work, we mention the three following publications.

- Note on the common tangents of a cubic and one of its sextactic conic [*BCMS*, 40 (1948) 135-139].
- Circles of double contact of a bicircular quartic [*BCMS*, 40 (1948) 207].
- Note on a circular cubic with a real inflexion at infinity [*BCMS*, 42 (1950) 73-81]

The main theme of the first of these papers is a general discussion on the common tangents of a bicursal cubic and one of its twentyseven sextactic conics. In the first part of the paper treatment of a general type (based on the use of homogeneous or projective coordinates) for finding the common tangents of a bicursal cubic Γ and an arbitrarily assigned conic (which may or may not be related to Γ) is considered. The second part concerns itself specifically with the common tangents of a (bicursal) cubic and a sextactic conic.

As regards, the second publication, as the name implies deals with certain novel trails of a bicircular quartic with special reference to its aggregate of circles of double contact.

It may be noted that, bicircular quartics are, broadly speaking, classified as bicircular or unicircular, according as their deficiency (or genus) is 1 or 0. Later, Dr. Haridas Bagchi also discussed by vector methods the general theory of affine transformations in Euclidean Space of three dimensions. Dr. Bagchi also wrote a treatise entitled *Course of geometrical analysis*. In later years Dr. Bagchi had done a lot of work on Spherical Functions and Functional Analysis, but that it is not within the framework of this article.

Surendra Mohon Ganguli (1881-1931) of the Department of Pure Mathematics also worked in another branch of Geometry. In 1922, he obtained his D.Sc. degree from the University of Calcutta for his thesis entitled "Analytical geometry of hyper spaces". He contributed the following two papers

- "On angle-concept in n-dimensional geometry" [*BCMS*, 1918].
- "On the geometry of four fold".

He also authored a treatise on *Theory of Plane Curves*, Vol. 1and 2 in 1928–1932.

Apart from these major contributions in the field of Geometry, some of Syamadas Mukhopadhyay's students also made notable contributions. Amongst them Prof. Gurudas Bhar a very eminent teacher of Presidency College, under the guidance of the brilliant geometer Prof. Syamadas Mukhopadhyay, published two notable research papers.

- "Geometrical construction for the limiting centres of a cubic" [*BCMS*, 12 (1920-21) 1-4].
- "The Osculating conic in homogeneous co-ordinates" [*BCMS*, 12 (1920-21) 85-104].

No article on geometrical research in Calcutta University would be complete without the mention of **Prof. R.N. Sen** (1896-1974). He made extensive contributions in the field of Differential Geometry. He was a student of Pure Mathematics of the University of Calcutta. In 1933 he joined the Department as a lecturer and became the Hardinge Professor of Pure Mathematics in 1954. Prof. Sen studied the simplexes in n-dimensions at the begining of his research career and his first paper entitled

- "Simplexes in n-dimensions" [*BCMS*,18 (1926) 33-64]

was the resulting output. His other papers published during that time also dealt with hyper surface. Some of his publications between 1928 –1931 need to be mentioned here.

- "Infinitesimal analysis of an arc in n-space" [*PEMS* (1928) 149-159]
- "Spherical simplexes in n-dimensions" [*PEMS*, Ser 2, 2, part I (1930) 1-10]
- "On curvatures of a hyper-surface" [*BCMS*, 23 (1931) 1-10]
- "On rotations in hyper-surfaces" [*BCMS*, 23 (1931) 195-209]
- "On one connection between Levi–Civita parallelism and Einstein's teleparallelism" [*PEMS*, Ser 2, 2, Part 4 (1931) 252–255].

This last research paper mentioned above aroused a lot of interest amongst famous geometers of his time. Prof. T. Levi–Civita of Rome was much impressed by his work.

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Abbreviations

Bulletin of Calcutta Mathematical Society- BCMS

Journal of Indian Mathematical Society- JIMS

Quarterly Journal of Pure and Applied Mathematics- QJPAM

Proceedings of Asiatic Society of Bengal- PASB

Proceedings of Edinburg Mathematical Society-PEMS

Bibliography

Banerjee, P.N., *Reforms and Reorganization – Hundred years of the University of Calcutta*, University of Calcutta, Calcutta, 1958.

Chatterjee, S.K., "Professor R.C. Bose: 1901-1987", *Calcutta Statistical Association Bulletin*, 37 (1988) 109-124.

Forsyth, A.R., *Differential Equations*, Macmillan and Co. Publishing House, London, 1960.

Indian Association for the Cultivation of Science, *A Century*, IACS, Calcutta, July, 1976.

IACS: Report of the Eleventh Annual Meeting, April 30, 1888.

IACS: Report of the Fourteenth Annual Meeting, April 30, 1891.

Mookerjee, Asutosh, *A Diary of Sir Asutosh Mookerjee*, Mitra and Ghosh Publishers Pvt. Ltd., 1992.