HISTORICAL NOTES

A GLIMPSE OF SOME RESULTS OF RAMANUJAN*

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Introduction

Mathematical wizard Srinivasa Ramanujan is regarded as 'a natural genius, in the class of Gauss or Euler'. He lived only a very short span of life (from 22.12.1887 to 26.04.1920), but his discoveries, of about two decades, are so great and glorious that they have been not only receiving worldwide recognition and acclaim but have inspired many to study, explore and extend his work.

He who ever has come in contact in any manner with him or with his mathematical work, has come under his wondrous spell. One cannot think about 'Ramanujan's life without some appreciation of Mathematics that he lived for and loved'. Profundity and richness in his works has been manifest even at the elementary level of mathematics. Here is a humble attempt to narrate a few of his works which will justify that his works at any level and his thinking process at any time may be considered by one and all to be always with exceptional touch.

Approximation of irrational number by using rational numbers

Here a narration is given with which many mathematicians were familiar, but only Ramanujan could observe its potentiality and exploited it

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to yield a very interesting and beautiful result.

   We know that successive terms of Fibonacci sequence $\{x_n\}_n$ are given by

$$1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ 55,\ \ldots\ldots\ldots..$$

where terms of the sequence are defined recursively by

$$x_1 = x_2 = 1,\ x_n = x_{n-2} + x_{n-1},\ \text{for } n > 2.$$

   It has been known since centuries that the newly formed sequence $\{x_n / x_{n+1}\}_n$, constructed from the terms of Fibonacci sequence , that is, the sequence

$$1/1,\ 1/2,\ 2/3,\ 3/5,\ 5/8,\ 8/13,\ 13/21,\ 21/34,\ 34/55,\ \ldots\ldots$$

is convergent and converges to the limit $(-1+\sqrt{5})/2$; and this limit is the positive root of the quadratic equation

$$1 = x + x^2.$$

   Ramanujan had the brilliant idea to reverse the whole procedure. He thought that, if we had the equation

$$1 = a_1 x + a_2 x^2 \ \ldots\ldots\ldots..(1),$$

(a_1, a_2 being rational numbers) and a sequence $\{x_n\}_n$ be formed in such a way that

$$x_1 = 1, x_2 = a_1,\ x_n = a_1 x_{n-1} + a_2 x_{n-2},\ \text{for } n > 2,$$

then question arises 'would it be true that the sequence $\{x_n / x_{n+1}\}_n$ converge to the root of the equation (1) ?' He found that it was, indeed, true. Thus a method of approximating an irrational number by rational numbers was found out by him. In fact, he went much further. Ramanujan stated that

   'Let x be such that it satisfies the equation

$$1 = a_1 x + a_2 x^2 + a_3 x^3 + \ \ldots\ldots\ \ldots\ \ldots\ldots..(2),$$

(a_1, a_2 , $a_{3\ldots\ldots\ldots}$ being rational numbers) and let us suppose that

   $x_1 = 1,\ x_2 = a_1$, and $x_n = a_1 x_{n-1} + a_2 x_{n-2} + \ldots.. + a_{n-1} x_1,$

   for $n > 2$,

then the sequence $\{x_n / x_{n+1}\}_n$ will tend to x as n tends to ∞.'

Interesting enough, he used this theorem to get an approximation of $\log_e 2$. Let us see how?

Let $\log_e 2 = x$; then we get $e^x = 2$

$\Rightarrow 1 + (x / 1!) + (x^2 / 2!) + (x^3 / 3!) + (x^4 / 4!) + \ldots\ldots = 2$

$\Rightarrow 1 = x + (1/2) x^2 + (1/6) x^3 + (1/24)x^4 + \ldots\ldots \ldots\ldots (3)$

Comparing (2) and (3), one gets

$a_1 = 1$, $a_2 = 1/2$, $a_3 = 1/6$, $a_4 = 1/24$, $a_5 = 1/120$, $\ldots\ldots$.

And hence the terms of the sequence $\{x_n\}_n$ are given by

$x_1 = 1$, $x_2 = a_1 = 1$, $x_3 = a_1 x_2 + a_2 x_1 = 1.1 + (1/2).1 = 3/2$,

$x_4 = a_1 x_3 + a_2 x_2 + a_3 x_1 = 1.(3/2) + (1/2).1 + (1/6).1 = 13/6$,

$x_5 = a_1 x_4 + a_2 x_3 + a_3 x_2 + a_4 x_1 = 1.(13/6) + (1/2).(3/2) + (1/6).1 + (1/24).1 = 75/24 = 25/8$,

Similarly, we get $x_6 = 541/120$ and so on.

Thus the terms of the sequence $\{x_n / x_{n+1}\}_n$ are given by

1, $1/(3/2)$, $(3/2)/(13/6)$, $(13/6)/(25/8)$, $(25/8)/(541/120)$, $\ldots\ldots$.

or, $\quad 1$, $2/3$, $9/13$, $52/75$, $375/541$, $\ldots\ldots$.

And the claim is that this sequence tends to $\log_e 2$ as n tends to α.

Indeed, $375/541 = 0.6931608$, whereas $\log_e 2 = 0.6930826\ldots\ldots$

This is really an astonishing result exploiting a well known formulations by looking to it in an exceptional way and obtaining a very excitingly beautiful result.

Relation of numbers raised to different powers

Ramanujan was fascinated with the relation of numbers raised to different powers. He not only found cubes of many numbers and tried to establish relations among them, but also putting them and their relations in an unbelievable way and could cite these at any time with the help of his extraordinary memory. Prof. Hardy rightly said, 'He (Ramanujan) could remember idiosyncrasies of numbers in an uncanny way. It was Littlewood

who said that every positive integer was his personal friend.' A very well known episode aptly justifies these remarks.

In England when Ramanujan was ill and lying in seek bed of a nursing house at Putney, Prof. Hardy had come to see him riding the taxicab bearing the number 1729. Arriving near Ramanujan he uttered the number and remarked, 'The number seems to me rather a dull one and that I hope it is not an unfavourable omen.' Indeed, 1729 is the product of just three prime numbers, namely, 7, 13, 19. But just listening to it Ramanujan's face glittered and he replied, 'No, it is a very interesting number. It is the smallest number expressible as the sum of two cubes in two different ways'. In fact,

$$1729 = 10^3 + 9^3 = 12^3 + 1^3.$$

Then and then Hardy enquired if Ramanujan knew the corresponding number of fourth power. Ramanujan thought just a while and answered, 'It must be a big one'. Actually, the number is a natural number of nine digits and it is 635318657, which is expressed as '$158^4 + 59^4$' or '$133^4 + 134^4$'. Incidentally, the number was discovered by Leonard Euler.

Here, one may note that 'the least number' as mentioned by Ramanujan meant 'the positive integer'; and 'two cubes' mean cube of two positive integer not of other kind of integers. If they denote not positive integers but simply integers, we would have the required least positive integer expressible as the sum of two cubes in two different ways as 91 in stead of 1729; because, $91 = 6^3 + (-5)^3 = 4^3 + 3^3$.

However, what is most remarkable of the episode regarding Ramanujan's exceptional power of memory and of recapitulation of relations among numbers is that there is no explicit mention of the number '1729' anywhere in his Notebooks or in any question sent to the *Journal of the Indian Mathematical Society*. However, relation equivalent to '$10^3 + 9^3 = 12^3 + 1^3$' can be found in a few places of his *Notebooks* (Vol. 2 p.266; p.387) and questions of the *Journal of the Indian Mathematical Society*. We mention here the following:

Question 661 (*JIMS,* 7, p. 119; 13, pp. 15-17; 14, pp.73-77).

Solve in integers

$$x^3 + y^3 + z^3 = u^6$$

and deduce the following :

$6^3 - 5^3 - 3^3 = 2^6$, $8^3 + 6^3 + 1^3 = 3^6$,

$12^3 - 10^3 + 1^3 = 3^6$, $46^3 - 37^3 - 3^3 = 6^6$,

$174^3 + 133^3 - 45^3 = 14^6$, $1188^3 - 509^3 - 3^3 = 34^6$.

Question 681 (*JIMS* 7, p.160; 13, p. 17; 14, p.73-77) .

Solve in integers

$$x^3 + y^3 + z^3 = 1$$

and deduce the following :

$6^3 + 8^3 = 9^3 - 1$,   **$9^3 + 10^3 = 12^3 + 1$,**

$135^3 = 138^3 = 172^3 - 1$, $791^3 + 812^3 = 1010^3 - 1$

$1163^3 + 11468^3 = 14258^3 + 1$, $6501^3 + 67402^3 = 83802^3 + 1$.

Besides these instances, there is a theorem in Ramanujan's *Notebooks*, which states that

'If $\alpha^2 + \alpha\beta + \beta^2 = 3\lambda\gamma^2$, prove that

$(\alpha + \lambda^2\gamma)^3 + (\lambda\beta + \gamma)^3 = (\lambda\alpha + \gamma)^3 + (\beta + \lambda^2\gamma)^3$.' .... (4)

It is interesting to observe that if we put $\alpha = 3$, $\beta = 0$, $\gamma = 1$, and $\lambda = 3$ in the relation (4) we get

$12^3 + 1 = 10^3 + 9^3$.

Really, what a display of superlative talent!

It may be mentioned here that at present a natural number, expressible as a sum of cubes of two positive integers in two different ways is termed as 'Ramanujan's number'. We give a few examples of this kind of numbers (first five) below:

$1729 = 10^3 + 9^3 = 12^3 + 1^3$

$4104 = 16^3 + 2^3 = 15^3 + 9^3$

$13832 = 24^3 + 2^3 = 20^3 + 18^3$

$20683 = 27^3 + 10^3 = 24^3 + 19^3$

$40033 = 34^3 + 9^3 = 33^3 + 16^3$

That Ramanujan was fond of relations of numbers of different powers can be established by going through his *Notebooks*. One can find many such examples. We give here only a few selected ones below:

$$1^3 + 6^3 + 8^3 = 9^3$$

$$18^3 + 19^3 + 21^3 = 28^3$$

$$3^3 + 36^3 + 37^3 = 46^3$$

$$1^4 + 2^4 + 12^4 + 32^4 + 64^4 = 2^4 + 39^4 + 44^4 + 46^4 + 52^4 = 65^4$$

$$4^5 + 5^5 + 6^5 + 7^5 + 9^5 + 11^5 = 12^5$$

$$5^5 + 10^5 + 11^5 + 16^5 + 19^5 + 29^5 = 30^5.$$

Continued fractions

Ramanujan's *Notebooks* contain almost 200 entries about continued fractions. Bruce C. Berndt rightly remarked, 'It seemed to us that no person in the history of mathematics possessed the skill that Ramanujan had in determining continued fractions for various functions or finding closed form of representation for continued fraction'.

Ramanujan not only discovered many beautiful formulae using continued fractions, but also displayed his extraordinary authority on the topic. In support of this I am going to narrate an interesting episode. Prasanta Chandra Mohalanobis pioneered many fields of science and was the founder of the famous Indian Institute of Statistics. He went to England for higher studies and was a student of Kings College, Cambridge. He had been at Cambridge for about six months when his mathematics tutor asked him, 'Have you met your wonderful countryman Ramanujan?' He had not yet met him, but heard of him. Soon Mohalanobis met Ramanujan and two became friends and very good friends too. One day being invited for lunch by Ramanujan, Mohalanobis reached Ramnujan's place at Trinity residence and saw him busy with cooking. Mahalanobis sat nearby and began to read a copy of the Strand magazine which published a number of puzzles to be solved by readers. One of these was related to the numbers of houses of two British army officers. Mahalanobis solved the puzzle in a few minutes and said to Ramanujan that he found out a solution of a puzzle by trial and error, but he was still wondering if there were more solutions. He told Ramanujan,

'As you are busy now, you can think about it after finishing your cooking'. Ramanujan replied, 'I am cooking vegetables with my hands and I can still think about it. Tell me the puzzle, there will be no problem'.

Mahalanobis said, 'It is the puzzle regarding the numbers of houses of two British officers lodging in a long road of Paris. A third officer came to meet them. He started his journey from beginning of the street and went on adding the numbers of houses on his way till he reached the first officer's house. After meeting the first officer and spending some time with him the third officer proceeded for the second officer. While going to meet the second officer he again added the numbers of houses in between the two houses. To his utter surprise, the third officer found that the total of the numbers is the same as he got before when he arrived the first officer's house. As soon as Mohalanobis finished his narration of the puzzle, Ramanujan asked him to note down an infinite continued fraction and said that from its convergents possible numbers for the first officer's house would be obtained and then the number of the house of the second officer would easily be obtained.

Mohalanobis was surprised and enquired, 'So, there are infinite number of solutions'. Ramanujan replied affirmatively. 'How do you get the solution?' was the immediate question. Ramanujan replied that after knowing the first solution, which was an easy one, and learning the number of house of the first officer it flashed instantly in the mind that it would be a continued fraction and it would begin with the number of the house of the first officer.

The continued fraction uttered by Ramanujan is given below:

$$6 - \cfrac{1}{6-} \ \cfrac{1}{6-} \ \cfrac{1}{6-} \ \cfrac{1}{6-...} \ \ldots\ldots\ldots(5)$$

The first, second, third, fourth, …….. convergents of the continued fraction (5) can be given as $6/1$; $6 - (1/6)$; $6-\{1/6-(1/6)\}$; $6 - (1/6 -[1/6 - \{1/6 - (1/6)\}];$ …….

that is, $6/1$; $35/6$; $6 - (1/35)$ or $204/35$; $6 - (204/35)$ or $1189/204$; …

So, the numbers of first officer's house, according to Ramanujan, would be 6, 35, 204, 1189, ………. .

The numbers of houses of first and second officers found out by Mahalanobis were 6 and 9. Because, $1 + 2 + 3 + 4 + 5 = 15 = 7 + 8$.

Now, let us examine when the number of the first officer's house is known how to find out the number of the second officer's house, which was said by Ramanujan to be easy to determine. Let us see really whether it is easy to us or not.

Let m and n be two natural numbers such that $n > m$. Let $(m+1)$ and $(n+1)$ be respectively the numbers of first and second officers' houses. Then by the condition of the puzzle, we have

$$1 + 2 + 3 + 4 + ……… + m = (m+2) + (m+3) + ……… + n$$

or, $1 + 2 + 3 + 4 + ……… + m = [\{1 + 2 + 3 + … + m + (m+1) + ……. + n\}$

$$– \{ 1 + 2 + 3 + …… + m + (m+1)\}$$

or, $2(1 + 2 + 3 + 4 + ……… + m) = (1 + 2 + 3 + 4 + ……… + n) – (m+1)$

or, $m(m+1) + (m+1) = n(n+1)/2$

or, $2(m^2 + 2m + 1) = n^2 + n$

or, $n^2 + n – 2(m+1)^2 = 0$ ………….. (6)

Now, (6) is a quadratic equation in n, whose positive root can be given by

$$n = [– 1 + \sqrt{\{1 + 8(m+1)^2\}}] / 2 ………..(7)$$

We observe that

when $m + 1 = 6$, $n+1 = 9$;

when $m + 1 = 35$, $n+1 = 50$;

when $m + 1 = 204$, $n+1 = 289$;

when $m + 1 = 6$, $n+1 = 1682$;

Really, what a insight and exceptional thinking of an ordinary situation!

It is to be noted that Ramanujan's investigation regarding continued fractions evolved by generalization of a particular continued fraction and it is termed as 'Ramanujan's continued fraction', which is given by

$$1/1 + x/1 + x^2/1 + x^3 / 1 + ………… \mid x \mid < 1$$

One interesting aspect about formulas on continued fractions discovered by Ramanujan is that when Ramanujan sent his discoveries in terms of formulae to Prof. Hardy along with his first historic letter, Hardy selected fifteen formulas from them as 'fairly representative'; and three of these were about continued fractions. Hardy's remarked that these 'defeated me completely. I had never seen anything like them before'. Really, these were 'on a different level and obviously both difficult and deep'. Here I just mention one of them below:

If $u = x\ /1+\ x^5\ /1+\ x^{10}\ /1+\ x^{15}\ /1+\ \dots\dots\dots$

and $v = x^{1/5}\ /1+\ x\ /\ 1+\ x^2\ /\ 1+\ x^3\ /\ 1+\dots\dots.,$

then $v^5 = u\ \{(1 - 2u + 4u^2 - 3u^3 + u^4\ )\ /\ (1 + 3u + 4u^2 + 2u^3 + u^4)\}.$

Highly composite numbers

Ramanujan's memoir entitled 'Highly composite numbers' appeared in the *Proceeding of London Mathematical Society* [2, XIV (1915), pp. 347–409] and for this long paper (containing 269 equations) Ramanujan was awarded B.A. degree 'by research' by Cambridge University in 1916. It is interesting to note that incidentally it was the longest paper of all the papers ever written by Ramanujan. About this memoir Hardy's remark is noteworthy. He remarked, 'There can be no questions to the extraordinary insight and ingenuity he (Ramanujan) has shown in treating it, nor any doubt that his memoir is one of the most remarkable published in England for many years'.

Let n be a natural number and d(n) denote the number of divisors of n. For two natural numbers N and $N^/$, Ramanujan in his above mentioned paper writes, 'A number N may be said to be a highly composite number, if $d(N^/) < d(N)$ for all values of $N^/$ less than N. It is easy to see from the definition that, if N is highly composite and $d(N^/) > d(N)$, then there exists at least one highly composite number M, such that

N< M ≤ $N^/$.'

It is known to us that for a natural number n,

$n = p_1^{n_1} \times p_2^{n_2} \times p_3^{n_3} \times \dots\dots\dots \times p_m^{\ n_m},$

where $p_1, p_2, p_3, \dots\dots\ ,\ p_m$ denote different prime numbers and $n_1, n_2, n_3, \dots\dots\ ,\ n_m$ denote different natural numbers.

Ramanujan proved that ' If n is a highly composite number then p_1, p_2, p_3, ………. , p_m are successive prime numbers and $n_1 \geq n_2 \geq n_3 \geq$ ………$\geq$ n_m and if n_m is the last index then $n_m = 1$ except for 4 and 36'. He also stated that

$$d(n) = (n_1 + 1)\ (n_2 + 1)\ (n_3 + 3)\ \text{…….}\ (n_m + 1).$$

In his paper Ramanujan discovered first 103 highly composite numbers and gave a list of them with their d(N) and d(d(N)). What an exceptional calculating ability!

First 15 highly composite numbers, taken from the list, are given by

2, 4, 6, 12, 24, 36, 48, 60, 120, 180, 240, 360, 720, 840, 1260.

The 103rd highly composite number given by Ramanujan is 6746328388800 and

$$6746328388800 = 2^6 \times 3^4 \times 5^2 \times 7^2 \times 11 \times 13 \times 17 \times 19 \times 23.$$

Here, $d(N) = 10080 = 2^5 \times 3^2 \times 5 \times 7$ and $d(d(N)) = 72 = 2^3 \times 3^2$ and it is interesting to note that 10080 is 20th highly composite number.

Ramanujan stated and proved many beautiful properties of highly composite numbers. Here, I mention only one of them. The statement of the property is as follows:

'Two consecutive highly composite numbers are asymptotically equivalent, that is, if x and y be two consecutive highly composite numbers then x~y; that is, (x/y)→1.'

Mathematical Amusement

To Ramanujan mathematics was only thing of concern and every thing of occupation. He never forgot to show his sense of humour and fun here. In his *Notebooks* he posed a problem as the following:

'Determine natural numbers a, b, c, d ……. and p, q, r, s, …… such that the following two relations are satisfied simultaneously :

$$a + b + c + d + \text{……..} = p + q + r + s + \text{…………...(8)}$$
$$a^a + b^b + c^c + d^d + \text{…….} = p^p + q^q + r^r + s^s + \text{………….(9)'}$$

It appears that it is quite difficult to determine natural numbers a, b, c, d, ... and p, q, r, …….satisfying both the relations (8) and (9). But Ramanujan tells us that the problem is not really difficult. He says that first find out the numbers satisfying (9), which is quite easy to find out and then manipulate it to get the relation (8). According to Ramanujan, for a solution of (9), we may very easily have the following relation

$$2^2 \cdot 6^6 = 3^3 \cdot 3^3 \cdot 4^4. \ \ldots\ldots\ldots\ldots\ldots \ (10)$$

This may appear very easy to Ramanujan, but certainly not so easy at all. Even this solution does not satisfy the relation (8); because the sum in the left hand side of (10) is 8 (= 2 +6) and sum on the right hand side is 10 (= 3+3+4). Ramanujan prescribed that this difficulty could easily be removed by introducing two factors 1^1 and 1^1 in the left hand side, which would not affect (9) but satisfy (8). We would get then

$$1+1+2+6 = 3+3+4,$$

and $$1^1 \cdot 1^1 \cdot 2^2 \cdot 6^6 = 3^3 \cdot 3^3 \cdot 4^4 \ .$$

Really, presence of factor 1^1 in any number does not create any difficulty to have the desired solution of the problem. Indeed, any solution of (9) yield a solution of (8) by introducing required number of factors of 1^1.

A few more examples are given below of which first two are taken from Ramanujan's *Notebooks*:

(i) $1^1 \ 3^3 \ 12^{12} \ 20^{20} = 5^5 \ 15^{15} \ 16^{16}$,

where $1+3+ 12+ 20 = 5+ 15+ 16$.

(ii) $1^1 \ 4^4 \ 20^{20} \ 30^{30} = 6^6 \ 24^{24} \ 25^{25}$,

where $1+ 4+ 20+30 = 6 + 24+25$.

(iii) $1^1 \cdot 1^1 \cdot 1^1 \cdot 1^1 \cdot 1^1 \cdot 1^1 \cdot 1^1 \cdot 1^1 \cdot 6^6 \cdot 10^{10} = 3^3 \cdot 3^3 \cdot 4^4 \cdot 4^4 \cdot 5^5 \cdot 5^5$,

where $1+1+1+1+1+1+1+1+ 6+ 8 = 3+3+4+4+5+5$.

(iv) $1^1 \cdot 1^1 \cdot 1^1 \cdot 1^1 \cdot 1^1 \cdot 1^1 \cdot 1^1 \cdot 1^1 \cdot 8^8 \cdot 12^{12} = 2^2 \cdot 2^2 \cdot 4^4 \cdot 6^6 \cdot 6^6 \cdot 8^8$

where $1+1+1+1+1+1+1+1+ 8 +12= 2+2+4+6+6+8$.

Really, what a thinking of an exceptional genius!

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