Indian Journal of History of Science, 48.4 (2013) 535-581

GENESIS AND EARLY EVOLUTION OF DECIMAL ENUMERATION: EVIDENCE FROM NUMBER NAMES IN *ṚGVEDA*

Bhagyashree Bavare* and P P Divakaran**

(Received 26 December 2012; revised 08 July 2013)

The origins of decimal enumeration are sought in the number names found in the *Ṛgveda*, through a study of the grammatical rules that go into their formation. Enumeration requires the choice of a base. It is suggested here that, once such a choice is made, written (symbolic), oral (nominal) or other forms of counting are different concrete realisations of an abstract 'place-value principle'. In the decimal (base10) nominal realisation as found in India beginning with the *Ṛgveda*, the different 'places' (ones, tens, hundreds, etc.) are marked by giving essentially arbitrary names to the powers of 10, just as the 'atomic' numerals 1 to 9 are assigned arbitrary names. For purposes of counting, the nominal realisation is as good as the written positional notation. Since any compound number can be expressed in terms of the atomic numbers and the powers of 10 by the operations of addition and multiplication, the name of every number can correspondingly be constructed by combining the names of atomic numbers and powers of 10 through the application of the grammatical rules of nominal composition appropriate for these two mathematical operations. The *Ṛgveda* is very rich in the names of compound numbers, the largest being just under 100,000. Apart from two or three cases of ambiguity, their grammatical analysis, in conformity with the rules of nominal composition of vedic Sanskrit, results in a unique association of a precise number to each name. Our main conclusion is that, much before the occurrence of written numbers in India, the *Ṛgveda* already provides very strong evidence of a mastery of the principles of decimal enumeration. Related topics discussed include the possible recognition of the unboundedness of counting numbers in early ritual texts and the reverse influence of these principles on theories of the structure of language, especially in the *Vākyapadīya* of Bhartṛhari.

*Department of Sanskrit, University of Mumbai, Mumbai 400098
**NCBS-TIFR, GKVK Campus, Bellary Road, Bangalore 560065; Email: ppdppd@gmail.com

Key words: Based numbers, Place-value principle, Oral culture, Atomic and compound numbers, Nominal composition, Numbers and grammar

1. General Introduction

Our ability to count, to quantify precisely the number of elements in a finite set, has become such a routine skill that we no longer wonder about the principles behind it. These principles are of course encoded in the place-value or positional notation for numbers.[1] That in turn requires a *base*, a unit in terms of which to 'measure' numbers, to be chosen. The choice of 10 as the base and the use of the resulting decimal place-value counting procedure are now universal – to such an extent that we have to make an effort to remember that, in many parts of the world, counting decimally is a skill acquired very recently on a historical time scale. Europe learned about decimal place-value reckoning certainly by the early 13th century AD (but probably earlier [Gu]) from contacts with the Arab world and called the number notation based on it the Arabic numerals. The Arabs themselves named them Hindu numbers, having forgotten that the first positionally expressed numbers were written down (with 60 as base) in their own cultural homeland many centuries before the earliest Indian records of the decimal system can be firmly dated. A comprehensive historical summary of the spread of decimal numbers in their written form from India into other parts of the world will be found in R.C. Gupta's survey [Gu].

In Europe, the introduction of the decimal system caused a revolution, even though a very slow one. As late as in the 16th century, arithmetic was done even in such computation-intensive scientific endeavours as cartography by methods not very different from those employed by Ptolemy of Alexandria. Later still, in the 1660s, Isaac Newton was referring to the decimal system as "the new doctrine of numbers" even as he took it as the model for the algebraic meaning and manipulation of infinite series.

The explosive growth of mathematics in post-Renaissance Europe, in all its aspects including astronomy and physics, is universally attributed to the discovery of the source of Europe's intellectual heritage in classical Greece. That discovery resulted in the elevation of geometry as the purest form of mathematics and Aristotelian axiomatism as the surest guide in all rational enquiry. But a fuller picture will also acknowledge the powerful influence of decimal counting and will place, alongside Descartes and the later Newton of the *Principia*, the many practitioners of arithmetic and the pioneers of algebra such as Viete, and indeed Newton himself, the young Newton of 1665-1675 casting about for a rationale for what he wished to do with infinite series. They paid not much attention to logical formalism or the axiomatic method, turning their hand to whatever tool met their needs – Newton writes of "analogies", "interpolation", etc. as some of the means by which he arrived at mathematical insights. Open as his acknowledgement was, the true measure of the influence of the "new doctrine of numbers" is the tremendous progress that arithmetical and algebraic mathematics made in Europe in the 18th and 19th centuries.

In the last decades of the 19th century, European mathematics closed the circle; there was finally a convergence of the freewheeling but productive methods of the previous two centuries and the ideal of logical rigour as embodied in the axiomatic methodology. As far as the concerns of this article go, the most interesting product of this unification was the axiomatisation of the notion of natural or counting numbers by Peano in 1889, creating thereby an impeccably Aristotelian logical foundation for arithmetic some thousands of years after it first became part of human thought.[2] But arithmetic had not in the interim waited around for someone to come and say that it was alright to add and multiply and so the chief immediate consequence of the new development was the growth of a specialised domain of mathematical logic concerning itself with foundational issues.

In India, the enterprise of acquiring and validating knowledge had an entirely different kind of foundation. From very early times, Indian savants rejected any form of axiomatism as a basis for rigorous thought. We have evidence from the time of the later *upaniṣads* (ca. 6th c. BC) onwards that they were a sceptical and questioning lot; little was considered self-evident and notions such as first causes and first principles which were by definition beyond questioning had no appeal for them.[3] It is in this context that our enquiry into the origins of decimal enumeration and its implementation in the earliest Indian textual material that can be read, the *Ṛgveda*, has to be set. The foundations of Indian science did not much change over the centuries and, throughout their long history, numbers were invariably thought of as being fully defined in relation to the base 10.[4] That is an advantage for us in that we can call up evidence from later material, some as late as the 16th century, to throw light upon questions on which the early texts are silent.[5]

More directly relevant for this study is another special circumstance, that the early vedic literary culture was dominantly if not exclusively oral. The demands made of oral and written media in communicating quantitatively precise ideas are very different, a theme which will keep coming up in this article. A written positional notation for numbers as on the Babylonian tablets is complete and well-defined in itself, subject only to one convention, that of the order in which the atomic numerals[6] constituting the individual entries of a number are written as the place or position goes up. A number or a simple sum written positionally conveys no more and no less than that number or that sum; it needs, if clearly enough presented, little support from words.

When it comes to number identity in an oral culture, the first goal will be to create a systematic method of nomenclature, articulated sound-patterns, which can stand on its own, i.e., which is as precise and unambiguous as the written representation. Ideally, such a goal is attainable. The first step is to name the atomic numbers arbitrarily just as their written symbols are arbitrary: in English, one, two, . . . nine. Once that is done, the most economical way to proceed is to invent names for each 'place' (each power of the base), ten, hundred, thousand, etc. These various names have to be combined, in more ways than one, to make up names for every number and there must be rules governing the ways the names are combined. There are only two such rules needed in an ideal naming system, one for multiplying the atomic numbers by the base and its powers and the other for adding up the resulting numbers, e.g., (1947=) $7 + 4 \times 10 + 9 \times 100 + 1 \times 1000$ = one thousand nine hundred and forty seven. These two rules can then be used recursively to create names for all numbers, the only fresh input required being names for higher and higher powers of the base. The rules of formation cannot be free from the structural features, both semantic and syntactical, of the language in which the naming and combining take place. An oral enumeration thus draws upon the resources of its ambient language to meet its quantitatively precise needs and, in turn, influences that language by the demands of those needs.

But languages are not designed by mathematicians or logicians, not even, as Patañjali observed long ago, by grammarians. Accommodating the terminological demands of counting was only a small part of the very many functions that vedic Sanskrit, like any natural language, was asked to perform. And in this, i.e., meeting the restricted needs of a rational number-naming paradigm, the primary objective should obviously be semantic clarity: which number precisely does a given number name designate? How this specific aim is achieved is thus subsumed in the general framework of rules, systematised long after the vedas were composed, for the formation of nominal compounds and the attendant phonetic transformations. But even in a syntactically well-regulated language like Sanskrit there is a degree of flexibility in the application of these rules;[7] an ideal, uniformly rule-based system that assigns a unique name to every number is therefore not to be expected. This is one of the ways in which the oral differs from the written system, but not the most important. That is because what is absolutely essential is that the association of numbers to names be unique – not depending on minor, usually but not always grammatical, variations in the names – rather than that the names themselves be unique. In mathematical language, while the map from numbers to names may be one-to-many, the inverse map from names to numbers must be well-defined. The main conclusion from our examination of the number names in *Ṛgveda* will be that this criterion is met in almost all instances despite the same number often having different verbal expressions beyond the variations imposed by the dependence of the compounding rules on gender, number, case, etc. of the words in question.

The *Ṛgveda* is astonishingly rich in words and phrases relating to numbers, around 3000 of them, many large enough to exhibit the fundamental characteristics obligatory for an effective orally expressed decimal system. After an account of the general framework in which to fit the data we can mine from it, we look at a substantial sample of them in enough detail to establish that all aspects of these principles were perfectly well understood at the time of its composition. In doing this we have not paid particular attention to possible changes in the way the same number or similar numbers are formed as we get to the later *maṇḍalas* 1 and 10; superficially at least, there seems to have been no such evolution. Indeed, vedic number names remained current for a very long time afterwards and, with minor changes, remain largely current today in most north Indian languages.

The way a predominantly oral language, in the present instance vedic Sanskrit, responds to the specific needs of science, in particular mathematics, has been written about before ([St2], [St5],[Fi]). The scope of the present work is narrower and does not cover, except to the minimum extent necessary, the many linguistic and grammatical issues that have to be addressed in any broad-based study of the way Sanskrit accommodates the needs of the sciences. Our objective is the limited one of assessing how well the logical requirements of an unambiguous number-naming system are mastered in the *Ṛgveda*. In other words, our concern is with the one question: how effectively are the rules of nominal composition utilised to construct, if not the ideal number-naming system that we evoked earlier, a satisfactory approximation to it? This is the question that we address in sections 6 and 7 (which therefore form the core of this paper).[8] To put the question and the answer that emerges in context, we have thought it useful to go over, in non-technical language and without any intent of being definitive, the elementary logical and arithmetical principles that underlie place-value enumeration. We will see that every function of a written positional notation as regards enumeration is performed as well by the number-naming rules that can be extracted from the *Ṛgveda*. Indeed, we shall argue that it is proper and useful to speak of an abstract place-value principle (and a concomitant place-value algorithm) of which the written and the oral expressions, among others, are representations in a precise meaning. Such a general viewpoint not only brings out the structural character of place-value enumeration but will also, we hope, act as a counterweight to the general tendency to think of it as exclusively tied to symbols and writing.[9]

The number naming rules are obviously of no use in apprehending numbers less than the base, the atomic numbers. Their names are entirely arbitrary and the association of names and numbers a matter of usage and familiarity, in other words a question for cognitive science. For this reason, we have made an attempt to trace (section 3), following [St5] and in the Indian context, the possible cognitive roots of quantitative enumeration whose evolved form is reflected in the decimal nomenclature analysed in sections 6 and 7.

That place-value enumeration is an absolute must for all mathematics, arithmetic to begin with, was understood early in India. Arithmetic hardly figures in the *Ṛgveda*; that should not be surprising in a poetical work. But the *Śulbasūtra* manuals (see the annotated English translation of Sen and Bag ([SB]), the earliest of which are dated perhaps 400 years after the compilation of the *Ṛgveda*, contain good evidence of a very competent understanding of the arithmetic of whole numbers and fractions in the midst of its geometry. Geometry and numbers remained closely linked throughout the history of mathematics in India, culminating finally in the development of infinitesimal calculus in the work of the mathematicians of the Nila (Kerala) school ([Di2]). Beyond that, creative generalisations of the place-value principle led to new algebraic concepts (polynomials and power series) and to new methods of proof (mathematical induction). The invention of calculus drew, in addition, on the idea of (numerical) infinity ([Di2]) of which the earliest intimations are present already in the ritual literature immediately following *Ṛgveda*. These aspects, fascinating as they are, are only tangentially referred to here. What we have touched on instead, here and there, is the role played by the decimal place-value paradigm in other areas of Indian thought by reference to Bhartṛhari's epistemology of language as an illustration. This reverse influence of the power of decimal numeration on grammar and language has not always been given its due.

A first account of some general aspects of decimal enumeration forms part of [Di1] which also looks at the question of possible external influences on its genesis. A summary of the basic relationship of number names with grammatical rules can be found in [Ba].

2. Measuring Numbers: Bases and the Place-Value Principle

The earliest indications of vedic numeracy lie probably in those passages of the *saṃhitās* that refer to matching or comparing numbers, of which we give several examples later in section 4. This is not counting as we use the expression commonly but a means of establishing the equality of two numbers without knowing either, i.e., of deciding, in modern logical terminology, when two finite sets have the same cardinality. It is the exact discrete counterpart, in a physical sense, of the process by which we determine when two objects weigh the same by placing them in the pans of a balance – counting is after all the most primitive measurement we can make in the physical world.

But in counting unlike in other measurements, early civilisations quickly progressed from such one to one comparisons to what would appear to be a method of absolute measurement by the use of a fixed number such as 10 as a unit of measurement.[10] That such an absolute sense of the magnitude of a number can be acquired by the use of a unit like 10 is to an extent an illusion, born out of our long and constant familiarity with the decimal system of enumeration. This is a fact easily verified by anyone trying to make quick sense of numbers written in relatively unfamiliar bases; for example, the string of 1s and 0s that make up the binary representation of even a moderately large number or, at the other extreme, the Babylonian sexagesimal representation of a number requiring more than two or three places to accommodate, will cause most people to convert to decimals to make quantitative sense of them. In principle, the choice of 10 as the unit for measuring numbers is no less arbitrary than the choice of the standard kilogram as the unit of mass.

The choice of a unit is only a first step. To be told that a number consists of a large number of 10s with a few (less than 10) left over rather than of approximately ten times as many 1s only postpones the difficulty one step. The key idea is the recursive one of applying the same procedure over and over again. Suppose a number N of objects leaves behind n_0 of them when as many multiples of 10 as possible, say N_1, are removed; this means that $n_0 < 10$, i.e., n_0 is an atomic number. If $N_1 < 10$, the process terminates. If not, from the number N_1(of multiples of 10) remove as many multiples of 10 as possible leaving behind $n_1 < 10$ multiples of 10, so $n_1$is an atomic number. Repeat the process. In more formal arithmetical language, the procedure amounts to the following. Divide N by 10 with quotient N_1 and remainder n_0: $N = N_1 \times 10 + n_0$; next divide N_1 by 10 with quotient N_2 and remainder n_1: $N_1 = N_2 \times 10 + n_1$ so that $N = N_2 \times 10^2 + n_1 \times 10 + n_0$; next divide N_2 by 10 and so on till, in k steps, we find $N_k = n_k$ itself to be less than 10. What this has achieved is to represent N as

$$N = n_k \times 10^k + n_{k-1} \times 10^{k-1} + \ . \ . \ . + n_1 \times 10^1 + n_0 \times 10^0$$

with each of the numerals $n_0, n_1, \ . \ . \ . \ , n_k$ less than 10. More concisely, the number N is completely characterised by the ordered set of atomic numbers:

$$\mathrm{N} = [n_k n_{k-1} \ . \ . \ . \ n_0],$$

which, except for the square brackets, is the conventional way of writing numbers.

Obviously, there is nothing special in principle about the base 10 – in practice, the base does make a certain difference ([Di1]) – in all this; any number b other than 1 would have done just as well. And for any choice b, it is equally obvious that every number N can be represented in this way,

$$N = n_k b^k + n_{k-1} b^{k-1} + \ldots + n_0,$$

and that the representation is unique, i.e., for any N, its place-value entries $n_1, n_2, \ldots, n_k$, in order, are uniquely fixed; conversely an ordered set of atomic numerals uniquely determines a number of which they are the place-value entries. It is this object, the set of natural numbers together with a choice of base b in terms of which each of them can be represented by a unique ordered set of natural numbers $n_0, n_1, \ldots$ (depending on the choice of the base), each less than b, that we call natural numbers with a base or, simply, based numbers; the procedure by which the place-value representation of a number is established we shall refer to as the *place-value algorithm*– it is no more than the recursive application of what is known in modern arithmetic as the division algorithm. The early enumerators would not have thought of the algorithm and the principle behind it as formally as we do, but that cannot in any way devalue the fact that they had to have an intuitive appreciation of the principle involved in the process, including a grasp of the arithmetic of division with remainder.[11] The way the result of the algorithm is presented – symbolic (written), nominal(oral) or some other (see later for examples) – does not change the fact that the principle itself is abstract and independent of how it is implemented. Its conceptualisation, however vaguely, is a necessary prior stage that the pioneers of enumeration had to start from, whatever method they chose to express the final result. It is to emphasise the logical precedence of the idea of place value counting that we refer to this principle as the abstract place-value principle having, for any choice of base, various possible representations, symbolic, nominal, etc. We also mean thereby to de-emphasise the common association of place-value counting exclusively to written numbers.

The same principle, when adapted to a variable base, led to the definition of polynomials and power series as abstract algebraic objects in the work of the Nila school in Kerala (*Yuktibhāṣā* [TA],[Sa], chapter 6), a theme which is not pursued here.

As far back as we can go, which is of course not earlier than *Ṛgveda*, numbers in India, *saṃkhyā*, always and without exception, meant based numbers with 10 as the base. And, in a very strong sense, they are the foundation upon which all of Indian mathematics was recognised to rest. As *Yuktibhāṣā* expresses it in the first chapter, "That which is the particular study of numbers relating to enumerables (*saṃkhyeyam*) is mathematics". What the use of a base does first is to provide an unending series of markers, exponentially spaced (the powers of 10), enabling us to know how far we are from the first number which was generally taken to be 1. The place-value algorithm gives us the ability to know absolutely and exactly where in the endless sequence of positive integers any number, however large, lies, but only after we have a means of answering the question "how many?" when the number is less than 10.

Perhaps we should think of the development of basic arithmetical skills and the evolving mastery of decimal numbers as seen in the early vedic corpus as taking place in tandem, each reinforcing the other.

3. Counting Without Counting

The place-value measurement of numbers in whatever form it is expressed, we may be allowed to repeat, gives people a means of knowing precisely where a symbolically written or named number stands in the order of all numbers by providing regularly placed markers. The primary markers are the places, the powers of 10. In the interval between two successive powers of 10, say 10^m and10^{m+1}, we have a secondary set of markers dividing the interval into a series of multiples of 10^m, then a tertiary subdivision of the interval between two consecutive such multiples, say $l \times 10^m$ and $(l + 1) \times 10^m$, into multiples of 10^{m-1} and so on until we get to multiples of the base 10. Thus in the interval between the primary markers 100 and 1000 we have the secondary markers 200, 300,..., 900 and, between say 300 and 400, the tertiary markers 310, 320, ..., 390, the primary markers at one level becoming the secondary markers when we go to the next higher place. What about the logical final step, that of apprehending precisely the numbers upto10? Habituated to dealing with them over millennia as we are, most of us seem to have an almost innate faculty of grasping such small numbers exactly as they are evoked by a symbol or a name and may not even consider this a serious question. But how essential a role long familiarity plays in facilitating this effortless and apparently automatic recognition will be plain to anyone learning number names in a new language for the first time or memorising number symbols which are unfamiliar and unrelated to the ones he or she already knows.

Number cognition has been a fertile field for both experimental and theoretical research by psychologists and other cognitive scientists, their chief preoccupation being to advance or refute the notion that the ability to count is an innate human faculty (see [St5] for a summary and a useful pointer to current research). (By "number" is meant, once again, the cardinality of a set, not its expression by symbols or names, a confusion which is present in some of the cognitive work).If it is, it cannot extend to numbers which are reasonably large; that is a matter of adult human experience. The burden of our discussion so far is then that the use of a base solves the problem of number identity (recognition) *once we have an independent method of dealing with* (cognition) *'small' numbers, those which are less than the base*. We would like to suggest that the key to the vedic solution of this problem of counting small numbers may be extracted from an observation of Renou [Re], as amplified by Staal [St5]. The verbal root *khyā* has the meaning 'to see' in Vedic; prefixing *saṃ* will turn it into *saṃkhyā* (= to see together) in its derived connotation of counting and thence to 'number'. In his study of the hymns to Agni, Renou ([Re]) cites the line (*RV* 4.2.18) *ā yūtheva kṣumati paśvo akhyad* ... which he translates as "(Agni) has looked at (people) like (one observes) groups of cattle at the master of cattle ...", adding a note to the effect that the meaning 'counted' for *akhyat* that may be implied here is not anterior to (the somewhat later) *Śatapatha Brāhmaṇa.* Staal takes the implication as established already in *Ṛgveda* (and in fact inserts "or counted" after "looked at" in the translation). To impute to Agni, endowed with light and vision, the power to see all and hence the divine gift of being able to count ('take in at a glance') appears to us entirely natural.

To put these associations of faculties and functions on a firmer footing, we can turn first to the meaning of other words derived from *khyā* and, simultaneously, other instances of the connection of Agni with clear vision. A sampling from Monier-Williams' dictionary ([M-W]) turns up the following (partial) list of words formed by attaching prefixes other than *saṃ* to the root *khyā* from early texts (*Śatapatha Brāhmaṇa*, *Ṛgveda* itself and, stretching a bit, *Mahābhārata*). Words which directly signify the faculty of vision include

ākhyā (*ā* + *khyā*) = to behold, etc.

prakhyā (*pra* + *khyā*) = to see, to be seen or known or be visible, etc.

prakhya = visible, clear, bright, etc.

vikhyā (*vi* + *khyā*) = to look at, to shine, illumine, etc.

In addition, and consonant with the relationship of *khyā* with *saṃkhyā*, such words can also have meanings which may not directly signify but derive from 'seeing' and often imply a prior act of seeing. Examples (some of the words are in common with the previous list, which is to be expected) are:

khyā = to be named, to be known, etc.

ākhyā = to tell, to communicate, etc.

prakhyā = to announce, proclaim, extoll, etc.

vyākhyā (*vi* + *ā* + *khyā*) = to explain in detail, etc.

vyākhyā = explanation, paraphrase, etc.

prakhyāta = known, celebrated, recognised, etc.

The connection of these words to 'seeing' requires no great imagination to establish; most of them can be derived from 'to make (others) see (what one has seen)'.

More intriguing at first sight is the conflation of 'knowing' and 'naming' inherent in the meanings of some of the expressions above, the first instances of a synonymy that runs through the long history of Indian theories of knowledge. Thus the linguist-philosopher *Bhartṛhari* (5th-6th century CE) says: "All knowledge [of what is to be done (*itikartavyatā*) in this world] depends upon the word (*śabda*)" and, again, "There is no cognition (*pratyaya*) in this world in which the word (*śabda*) does not figure. All knowledge (*jñāna*) is, as it were, intertwined with the word." (*Vākyapadīya* 1.121 and 1.123, in the translation of Iyer ([Iy]); Biardeau ([Bi]) and Pillai

([Pi]) have very similar interpretations). Surely, this identification originally sprang from the orality of language; in the context of the present work, it is an essential ingredient of (oral) number nomenclature.

The other link, that of Agni with the power to see (and to make see) with clarity, occurs explicitly and often in the *Ṛgveda*. Here is a selection (the translations are those of Wendy Doniger O'Flaherty ([DO'F]), except for the last two lines which are our renderings):

idaṃ me agne kiyate pāvakā'minate ... (*RV* 4.5.6)

O! Agni who makes things clear ... .

... *samānamabhi kratvā punatī dhītiraśyāḥ* (*RV* 4.5.7)

Let our vision that clarifies ... reach him (Agni).

vi jyotiṣābṛhatā bhātyagnirāvirviśvāni kṛṇute mahitvā (*RV* 5.2.9)

Agni shines forth with a high light; by his power he makes all things manifest.

dhruvaṃ jyotirnihitaṃ dṛśaye kaṃ mano javiṣṭhaṃ patayatsvantaḥ (*RV* 6.9.5)

He (Agni or the Sun) is light firmly fixed for everyone to see.

yaḥ parasyāḥ parāvatastiro dhanvātirocate (*RV* 10.187.2)

Who from beyond the far shore shines intensely across the desert.

yo viśvābhi vipaśyati bhuvanā saṃ ca paśyati (*RV* 10.187.4)

Who observes and comprehends all things in this world minutely and properly.

The causal relationship between illumination and the faculty of vision and perhaps of counting – "comprehends ... minutely"– is brought out clearly in all these passages. Interestingly, they do not always distinguish sharply between Agni's divine power of seeing and the more naturalistic instrumentality in enabling others to see.[12]

Whether Agni was the first enumerator or not, the general idea of associating light and clear vision with the faculty of taking in (small) numbers at a glance is a plausible and attractive one. From an empirical point of view, the acknowledgement given to the problem of counting is even more fundamental: how could a capacity for the apprehension of numbers, quantitatively and precisely, be accounted for except as a divine gift or, more prosaically, a corollary of the gift of vision?

How small does a number have to be for this to be possible? It is reasonable to suppose that that will depend on external factors to some extent, in particular on any regular pattern that may be imposed on the 'things' being counted by means of visual, auditory or gestural (or, conceivably, other) cues. In looking for empirical guidance, we do not need to worry unduly about what cognitive scientists might have to say on the question of whether numeracy is an innate faculty. The issue does not concern so much the virgin minds of the very young as it does motivated and trained adults – at least the not so young, such as the *brahmacārin*, men of learning in the making, of the Indian tradition. So we content ourselves here by describing a casually chosen example of number (re)cognition from one field of traditional Indian learning and teaching, that of music, in place of elaborate experiments.

The rhythmic patterns (*tāla*) of Indian music are based on a few canonical cycles, each *tāla* consisting of a fixed number of beats, with the cycles organised in fixed subcycles. The most common *tāla*, *tritāla*, has sixteen beats (*mātrā*, the same term as is used to denote the duration, short or long, of a syllable in the systematics of Sanskrit prosody) in four subcycles of four beats each. The initiate is trained in keeping track of the rhythm with the aid of an external device such as marking the beats on the knuckle joints of the fingers but that skill soon evolves into an infallible, almost innate one might say, sense of where one is in the *tāla* cycle;[13] the cycle, repeated over and over again, helps in this by accenting certain fixed beats, say the ninth and the thirteenth in *tritāla*, as markers. It is virtually unheard of for a trained musician to miss the beat.

The antiquity of the organisation of the corpus of *tāla* is hard to establish with any certainty, but that their roots go back to the structured recitation of the vedas, especially the chanting of *Sāmaveda*, seems very likely. Even after discounting melodic embellishments as found in it (and the accented syllables already present in *Ṛgveda*), the format of the metrically organised *saṃhitās* is founded on adherence to rigorous prescriptions regarding the number and sequencing of syllables (*mātrā*), the text therefore demanding a subconscious counting of beats in its recitation. The most common meters in *Ṛgveda* have syllables varying in number from 8 to 12 in each *pāda* (foot, line). Once again, there are auditory cues in the chanting as indeed there are in the recitation of poetry in virtually all Indian languages to this day, and even a relative novice will quickly spot deviations from the correct count and the correct stresses.

It would seem then that a perfectly reasonable explanation for the ability to count can be put forward without getting enmeshed in debates about innate faculties, at least as far as adult humans are concerned. Not only have we strong indications of the human capacity, after training, to take in relatively small numbers accurately without actually counting; it would seem also that the limit for the magnitude of numbers so apprehensible is around ten, a few more or a few less depending on the conditioning of receptive minds. The evidence we have adduced is no more than suggestive. To this we can add the fact that even in the written sexagesimal positional notation of Mesopotamia 10 was assigned a distinctive symbol and played a special role, the role of a visual cue. With all this in mind, we suggest with some confidence that the main reason for converging on 10 as the base, perhaps settled after long subconscious experimentation – we have already noted that this can only have developed in parallel with the elementary arithmetic of small numbers – is precisely its easy tractability, not primarily the biological accident of humans having 10 fingers. We shall see later that in one of the oldest forms of gestural decimal counting in India the numbers 1 to 10 are ticked off not on individual fingers but on the joints of the fingers of one hand in a clever and well thought out sequence.

4. The Abstract Place-Value System and its Realisations

The progression from 'counting without counting' to a full-edged place-value system enabling accurate enumeration independent of the magnitude of the numbers being counted must have been slow and erratic. Among the possible early stages of this progression we have to include what may be called comparison-counting, i.e., determining the relative cardinality of two sets by a one-one correspondence. Ideally one would like to find examples in the *Ṛgveda* for such an evolutionary path having actually been taken, but such evidence is not easy to find or evaluate: the individual poems having been composed over a period of time preceding their redaction, it is impossible to know when any particular practice or grammatical usage gained currency, without very detailed and often uncertain philological investigation. Related to this is the tendency of archaic usages and archaic formulations of ideas to persist in a language long after functionally more efficient replacements for them have found wide acceptance. In that perspective, the fact that the *Ṛgveda* has, while glorying in a profusion of place-value numbers, also instances of number-matching need not be thought illogical. Thus we have the particularly evocative (*RV* 1.50.7):

vi dyāmeṣi rajaspṛthvahā mimāno aktubhiḥ

paśyañjanmāni sūrya

which O'Flaherty ([DO'F]) translates as: "you cross heaven and the vast realm of space, O sun, measuring days by nights", with its reference to measuring (*māna*) and no reference to either the number of days or the number of nights. These numbers (360 and 720 in a year) occur in two other verses from the same *maṇḍala*, 1.164.48 and 1.164.11 respectively. *Taittirīya Saṃhitā*, compiled slightly later but referring to rituals going back to an earlier time, is especially rich in passages comparing named numbers, e.g. (5.1.1) in which the mere fact of disparate sets having the same cardinality (four ladles of oblations, four feet of cattle, four quarters; eight offerings to Savitā, eight syllables of *gāyatrī*; etc.) seems to imbue them with a magical unity (see Keith's translation [Ke] for the whole passage) – it is as though the magic, the power, is inherent in the abstraction of the number, a power that unifies all sets of concrete objects sharing the same cardinality. Similar "harmonies of numbers" (in Gonda's felicitous phrase) from the vedic corpus, quite a few from texts later than the *Ṛgveda*, can be found in [Go1]. Then there is the very well-known cows-for-bricks passage, also from the *Taittirīya Saṃhitā* (4.4.11) which we quote partially (Keith's translation [Ke]): "May these bricks, O Agni, be milch cows for me, one, and a hundred, and a thousand ...(and so on) ... and *parārdha* (probably 10^{13})". This last passage is particularly interesting from the place-value perspective since the naming of powers of 10 has by now gone far beyond the *ayuta* (10,000) of the *Ṛgveda*. The residual persistence of cardinality-counting – the introductory line about matching numbers of cows with numbers of bricks – alongside the free employment of one of the essential ingredients of oral decimal counting, that of naming consecutive powers of 10, is striking.

Nevertheless, the transition from number comparisons to their measurement by means of a standard base, 10, must have been largely completed well before the composition of the earliest poems of the *Ṛgveda*; the abundance of complex (non-atomic) number names in all *maṇḍalas* argues for such a view. And that, as we have seen, is best thought of as a particular expression or representation of the idea of a based (decimal) number in the abstract.

Historically, there have been several different modes of expression of this abstract object, each rooted in the particular culture that gave birth to it and responding to a particular need. First and foremost is the written one, and not only because it is the earliest recorded (Babylonia, with 60 as base); its primacy was assured once writing became widespread as an element of literacy and even more so after the invention of printing by means of movable types. The representation of a number N by the linearly ordered sequence of symbols for the atomic numerals $n_k, \ldots, n_0$:

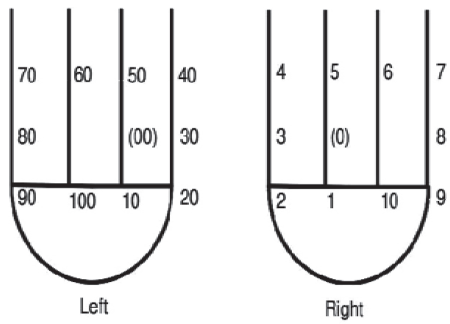
$$N \rightarrow [n_k n_{k-1} \ldots n_0],$$

fits in perfectly with the linear nature of writing and so is easily incorporated into written and printed text.

It is clear that the place-value system in a written representation cannot do without a symbol for zero. The early Babylonians did not have one, making do with a gap, an empty slot, ‘nothing’, to indicate that a particular position in a written number was unoccupied. An unwritten zero among positional entries can then cause confusion depending as such a practice does on the scribe’s adherence to accurate spacing. When there is a string of zeros, and especially when they occur at the right extreme of a number, the difficulty becomes serious: 1, 10, 100, ..., will all be represented by 1. The first modern students of the numerical tablets managed to sort out most such ambiguities from their context, but the experience surely played a part in the exaggerated importance given to the whole question of written representations of zero in some mathematical-historical circles; it is a far less critical issue in an oral tradition. (For the pre- and proto-history of the unwritten zero, see the articles of R. C. Gupta, M. D. Pandit and S. R. Sarma in [BS] and Staal [St7]).

In every respect other than the need for a zero symbol, however, the superiority of the written over other realisations is manifest. Apart from a degree of permanence that does not rely on memory, the fact that the written 'length' of a number is a logarithmically scaled (and hence compact and easily grasped) rough measure of its magnitude and that it serves as the primary marker are the most obvious advantages. Much more significant is the facility with which it lends itself to arithmetical operations. The potential of the place-value paradigm, itself born out of arithmetical impulses, to extend the empirically established and memorised rules of addition and multiplication of atomic numerals (there is no other way) to arbitrarily big numbers is virtually unrealisable without recording in some form the results of the intermediate steps. Devices like the abacus can be used for that function but we cannot doubt that the compactness and economy of resources of writing – all that is needed is a surface and an implement to scratch it with, sand spread on the floor and the tip of a finger for example – played a big role in turning arithmetic from an accountant's dark art into a skill as easily imparted as literacy itself. Even in the context of the Indian oral tradition, it is difficult to conceive that any but the simplest arithmetical manipulations could have been done without some means of recording, at the very least for temporary storage of intermediate results. The direct evidence we have for written arithmetic is, unfortunately, relatively late: the Bakhshali manuscript (see [Ha]) contains extensive written and drawn calculational schemata, but even its earliest plausible dating will not take us before the 3rd century AD. The references to *pāṭī*, a writing board, and to 'doing *rāśi*', a method of calculation using tokens placed in compartments representing places, are from much later times. But, despite the paucity of direct evidence, we have to wonder whether the arithmetic involving fractions that the earliest *śulbasūtras* (8th-7th centuries BC) ([SB]) describe in connection with geometric transformations could at all have been handled purely orally.

Before turning to the nominal realisation such as is found in the *Ṛgveda*, we deal quickly with one other mode of expression of the abstract place-value system, plausibly as old as the *Ṛgveda*. It is a method used by reciters of certain *mantras*, required to be repeated a certain number of times, to keep the count. The particular *mantra* we have in mind is the *gāyatrī* (*RV* 3.62.10) whose words, meaning and ritual significance are of no relevance here, only the fact that it has to be repeated a prescribed number of times (generally 108, on certain special occasions 1008). The reciter keeps track of the count by what we may call a gestural place-value method, using the finger joints of the two hands to mark the atomic numbers, with the right hand as the place of ones and the left as the place of tens. On the right hand, (some of) the joints are assigned the values 1 to 9 as shown in the stylised figure and counting is done by moving the tip of the right thumb over the joints, one step for each repetition of the *mantra*, starting with the initial position (0).



During the first complete cycle of the right thumb, the left thumb remains at (00). When the right thumb gets to 10, the left thumb is shifted also to 10; the right continues on to 1, 2, etc. till it reaches 10 again at which point the left thumb moves to 20, and so on. Eventually, when the left thumb reaches 100, the reciter takes one out of a bunch of ten blades of sacred grass with which he began the recitation and keeps it to one side. The two thumbs go back to their initial positions and the whole process is repeated till all the blades of grass have been shifted, marking the completion of a thousand repetitions.[14]

The reason for going over this, one of the most commonly practised brahmanic rituals, is that *maṇḍala* 3 from which the *gāyatrī mantra* comes is thought to be among the earlier portions of the *Ṛgveda*. There is no way of being sure that this particular method of counting is of equal antiquity but, just as surely, there is no way of ruling it out either. Given the extreme conservatism of brahmanical ritual practices, we would lean towards such gestural sequences accompanying the recitation of *mantras* being at least as old as the *śrauta* literature (c. 800 BC) that codified most rituals, with essential elements of them linked by the collective memory to a remoter past. It is in fact very likely that the structure of rituals in general, especially their repetitive and recursive content, has strongly influenced the foundations of the grammatical rules of vedic and later Sanskrit (for the case for these possible connections see [St3], [St6]) and decimal counting itself ([Di1]).

There are other realisations of decimal numbers in the Indian tradition, from much later, that should be mentioned here. The *kaṭapayādi* realisation associates every atomic number with one out of a set of syllables from the Sanskrit syllabary in an ingenious and flexible manner. What results is a representation of arbitrary (non-atomic) numbers by a string of syllables, meaningful words, lines and even whole verses, so as to facilitate their memorisation ([DS]). It is primarily a method of enumeration, not very practical for doing arithmetic. Computations are done in the *rāsī* realisation in which places are contiguous boxes marked on a piece of cloth (generally), occupied by tokens (generally cowrie shells) of the appropriate (atomic) number. Both these realisations are associated with Kerala.

5. The Oral or Nominal Realisation – Generalities

Early in the systematisation of place-value nomenclature, it must have been realised that the numbers 1 to 9 played a dual role in enumeration, first as the cardinality of sets containing 1 to 9 members and hence as entries in the place of ones, and next as entries in the places of tens, hundreds, etc. Considered as ordinals, therefore, the numbers 1 to 9 are special, they are the 'atoms' of which all numbers, however large, are composed.[15] One would have liked direct textual support for the hypothesis that the difference was understood if not formalised from early on, as it must have been. There appears to be no such support in the mathematical and astronomical literature but the following passage from Bhartṛhari (*Vākyapadīya* 1.87) provides a corroboration of the idea from a strikingly unexpected quarter:

yathādyasaṃkhyāgrahaṇamupāyaḥ pratipattaye

saṃkhyāntarāṇāṃ bhede'pi tathā śabdāntaraśrutiḥ

The key word here for us is *ādyasaṃkhyā*. Frits Staal translates it (in an isolated citation in [St1]) as "first numbers" and the whole stanza as "Just as grasping the first numbers is a means for the understanding of other numbers, even when different, so is hearing other words". Iyer in his translation of the whole work ([Iy]) gives the stanza the same general meaning, except for one crucial difference: *ādya* is rendered as "earlier" or "lower". We should first note that, for either meaning of *ādya*, this is already quite a remarkable statement. Not only is a distinction made between small ("first") numbers and large but, in a reversal of the conventional position, it is numbers which serve as a model for grammar rather than the other way round. To go beyond this general endorsement of the role number formation rules may have played in the epistemology of language, we can turn hopefully to the *vṛtti*[16] for a possible elucidation. The gloss on 1.87 goes as follows:[17]

> As the numbers one, etc..... serving different purposes are the means of understanding numbers like hundred, thousand, etc. and are considered a constituent part (*avayava*) of hundred etc., so the apprehension of a sentence is based on the precise meaning of words such as Devadatta, etc., the understanding of which is inherent (or implicit)....

Clearly, the two alternative readings, "first numbers" or "preceding or earlier numbers" lead to very different interpretations for the role "*ādyasaṃkhyā*" are intended to play: in the first reading they are our atomic numbers in a place-value context ("serving different purposes" depending on the place) while the second seems to imply an almost Peano-esque point of view, that a number is defined by all its predecessors. The gloss, with its stress on hundred and thousand, would appear to support the first alternative.[18]

Note also the insistence that the understanding of the precise meaning of 'constituent' words (Devadatta etc.) is inherent, exactly as the precise numerical significance of the atomic numbers is inherent.

Much later, we have a proper, unambiguous, formal enunciation of the dual function of atomic numbers in a 16th century text (in Malayalam prose) from Kerala, Jyeṣṭhadeva's *Yuktibhāṣā* ([TA],[Sa]).[19] The opening chapter entitled *saṃkhyāsvarūpam* ("The nature of numbers") has a concise but very instructive account of the principles behind the decimal place-value system. Here is our translation of the relevant passage:[20]

> That which is the particular study of enumerables (*saṃkhyeyam*) is mathematics (*gaṇitam*). The numbers from one to ten (written in words in the text) are like (or as if they were) *prakṛti* (that which occurs naturally, the original). Each of them, when multiplied by ten, that is [the multiples of ten] up to a hundred, are like their *vikṛti* (modification, variation, etc.)[21] The multiples by ten of these (the *prakṛti* numbers) will occupy one place higher than the place of ones. Next consider these [multiples

> of ten from ten to a hundred] as *prakṛti*; when they are multiplied by ten, the [resulting] numbers up to a thousand will occupy one place higher. In this way multiplication by ten of each of these [numbers] produces the numbers that follow, and their places are one higher at a time [for each multiplication by ten]. The names of the first eighteen places are ... (here follow the well-known verses 10 and 11 from Bhāskara II's *Līlāvātī*, naming the powers of 10 up to 10^{17}, beginning with *eka* and ending with *parārdha*). Thus if [we] endow numbers with multiplication and positional variation (*sthānabhedam*) there is no end to the *names of numbers; hence* (our italics, of course)we cannot know [all] the numbers themselves and their order. So, for practical purposes (*vyavahāram*), think as follows. The numbers one to nine [are] in the first place. Then the place of all of them multiplied by ten [is] the second. It is imagined to be to the left. The place of ones (*ekasthānam*), the place of tens (*daśasthānam*) and so on[are] their names. This [is] the nature of numbers.

Even in a work that abounds in mathematical insights of various sorts, this is an extraordinarily rich passage. The opening one-line definition "That which is the study of *saṃkhyeyam* is *gaṇitam*" testifies to the primacy of numbers in all of Indian mathematics. Then follows a succinct statement of the fundamental recursive principle on which the construction of decimal numbers is based – it is as though the wonder at the taming of numbers by the simple means of picking 10 as a unit of measurement has carried over undimmed across the centuries. The dual ('flower-in-the-garland') role of the numerals from 0 to 9 – technically more accurate than the textual 1 to 10 – as a natural corollary of the place-value principle is thus given its proper setting and importance.

Of equal value for us is the sentence following the power list from *Līlāvatī*, with its intimation of the infinitude of numbers and the consequent impossibility of naming them all,[22] and the declaration that, because they cannot be named, they cannot be known. This one flash of insight illuminates for us the central epistemic concern of orally literate cultures: how may one make an abstract 'thing' 'exist' except by recreating it in the mind as (the sound of) its name? As though to reinforce the message, we are asked to *imagine* (the verb used is derived from the Sanskrit *kalpanā*) that the place of 10s is to the left of the place of 1s. *Yuktibhāṣā* was written long after the positional writing of numbers became commonplace in India, and a thousand years after Bhartṛhari analysed the relationship between 'sound' and 'word' (*śabda*) and 'meaning', but we still see the mathematical mind resonating to signals from the distant but unforgotten past.

It is perfectly obvious that the numbers 1 to 9 can be assigned any name just as, in the written realisation, they can be assigned any symbol, subject only to ease of use and aural recognition. Usage will make the association of name to number robust and, in due course, people will have convinced themselves that they 'know' what each of them 'is' precisely; numbers will 'exist' and their understanding will become 'innate'. In a written realisation, the first choice of symbols for the atomic numerals is the only freedom; every subsequent number beginning with 10 has one and only one representation, once we adjoin a symbol for zero. In the oral or nominal expression on the other hand, we have to invent a name for ten, short of saying 'one zero' or 'zero one' and the same freedom exists for all powers of ten. The clumsy and slavish imitation of the written format implicit in the latter option could not have been exercised by the early decimalisers, not only because there probably was no writing at the time; there is also no evidence from the vedic corpus for a mathematical notion of zero ([BS, St7]), it was not needed. There is also a third option whose simplest illustration would be to designate 100 as 'ten tens' for example (a choice occasionally exercised in the *Ṛgveda*, as in *daśa sahasra* for 10,000) and more generally would involve fabricating a name for 10^k deriving from the name of the number k(much as in the use of the term *karaṇī* for the square root in the *śulbasūtra* ([SB]), *dvikaraṇī* = $\sqrt{2}$, *trikaraṇī* = $\sqrt{3}$).This would have needed a subtler appreciation of the mathematical notion of an exponent (k) than probably existed at the time. Whatever the reason, the option was not used. The overwhelmingly favoured strategy is that of giving new names to all powers of 10. The highest power of 10 so named, in fact the highest power of 10 occurring in the *Ṛgveda* (though not the largest number), is *ayuta*, meaning 10,000 as in all later texts.

In an ideal decimal nomenclature, there is no more freedom than that of naming the atomic numbers and the powers of 10. By ideal we refer primarily to two attributes. Most critically, the naming method must be rule-based and systematic in order to leave no ambiguity in the association of number to name. The rules must also be as few as will serve this purpose. These requirements are met if the rules, which are necessarily grammatical rules, are universal. The rule for naming the multiples of powers of 10 for instance must be independent of the two inputs, the power of 10 involved and the atomic number which is the multiplier. More formally, such a *multiplicative* rule will take two inputs, the names for n ($2 \leq n \leq 9$) and 10^m ($m \geq 1$), and return the name for $n \times 10^m$ as the output. It goes without saying that the result should fit into the grammatical framework of the ambient (spoken) language and it is this demand that most clearly distinguishes the nominal realisation intrinsically from the written one.

To complete the scheme, we need one more rule, one which will pinpoint, by its name, a particular number lying between two consecutive multiples of a fixed power of 10, e.g., a number between 10 and 20 or between 100 and 200 and so on. Here again it is elementary arithmetic which guides the naming paradigm. The only universally applicable arithmetical operation that achieves this end is addition – thus all numbers between 10 and 20 can be obtained by adding the atomic numbers to 10 (but cannot be by multiplication). The most natural linguistic equivalent is conjunction, (the name of) m and (the name of) n, which, like the mathematical operation of addition, is commutative, not sensitive to the order in which names are conjoined. This *additive* rule gets us to 19; 20 already has a name from the multiplicative rule after which the additive rule takes over again and so on, exactly as in the recursive construction of written numbers. The hierarchy of ideal number-naming rules is then: first invent names for all atomic numbers and for powers of 10; then set down a linguistic rule for multiplication of powers of 10 by atomic numbers and, finally, another linguistic rule for the number resulting from adding any of the numbers arising from the previous steps.

The above discussion of the fairly obvious systematics of a complete and efficient number-naming paradigm makes one thing clear. It is subsumed in the part of grammar that deals with combining words so as to create new ones with related connotations or serving related purposes (*prakṛti* and *vikṛti* again), but forms only a small part of the full spectrum of the rules for word combination that language as a whole has to accommodate. But no language is ideal in the sense of having a unique way of implementing a given functional requirement. That need not be a drawback as long as variations are recognisable as referring to the same general rule. With very few exceptions, the requirement will be seen to be met in the number names of the *Ṛgveda*.

6. Number Names in *Ṛgveda* – A First Look

It is useful to start by reiterating the main point of section 5. The practical execution of an ideal method of number nomenclature depends on two related conditions being met, one primarily logical and the other linguistic. The first is that whatever variant form the name of a general number may take should lead to no uncertainty in the exact identity of the number so (variously) denoted or described. Secondly, to be able to assert that this is so one must identify, on the basis of grammatical principles, the rules which govern the formation of names of numbers which are neither atomic nor powers of 10 in all their variations. These latter form part of the rules of nominal composition (which involve, generally, phonetic transformations at the junction of the words joined). Since the formal setting down of compounding rules postdate their manifestations in the *Ṛgveda* (and in vedic Sanskrit generally) of which it is, to a large extent, the systematisation, we are forced to rely on how later texts read the numbers and on what justification they gave for their reading. The continuity and the robustness of the tradition of memorisation is then a guarantee that we cannot go far wrong, just as the same continuity allows us to take the help of similarly late texts to validate the interpretation of mathematical ideas. Thus, in resolving ambiguities of interpretation, we have had to fall back upon the long line of grammarians and commentators beginning with Pāṇini and especially, in case of residual doubt, on Sāyaṇa's commentary. The recourse to later authorities is not always decisive however, as we shall see. One interesting feature that emerges is that the analysis of a number name often ends up as a description of the underlying arithmetical operation – indeed what we cite as number names are themselves sometimes phrases indicating the arithmetical process at work. This is not surprising once it is recognised that the vedas were the laboratory in which the grammatical tools that later evolved into settled rules were being forged. We should also add that the examples analysed in detail form only an illustrative sample of all the numbers that occur so profusely in *Ṛgveda*.

The first two stages of the naming methodology, namely the names of the atomic numbers and of the powers of 10, involve no linguistic considerations since they are arbitrarily assigned – the former because that is inescapable and the latter by choice, by not tying the name for 10^k to the name for k. The atomic numbers 1 to 9 occur freely in all the *maṇḍalas* in their unique names *eka,..., nava*; indeed the poets were so thoroughly conversant with them that the context-dependent variations of their names – for instance, the declensions of the first four of them, for which there is strong textual evidence – seem to have been already standardised.[23] These details can be relevant to our purpose – e.g., in the declension of the name for 200, see below – but we need not discuss the atomic number names themselves any further.

It is convenient at this point to make a useful terminological distinction. We shall call numbers which are neither atomic numbers nor powers of 10 *compound numbers*. This is basically an arithmetical notion; in the nominal representation they are also those numbers whose names cannot be picked arbitrarily but must be determined by the application of grammatical rules to the names of their constituent numbers. Among them will be purely *additive* and purely *multiplicative* compound numbers, of the form $10^k + 10^m$ and $n \times 10^k$, ($n < 10$) respectively, by means of which all numbers can be constructed. In most cases the grammatical rules are implemented in *Ṛgveda* through nontrivial linguistic and/or phonetic transformations on the constituent number names. In particular, additive compound numbers have names which either connote explicitly the operation of addition or result from them through transformations that are almost always unambiguous. But there also occur compound numbers in which the name of an atomic number (n) and that of a power of 10 (10^k) are simply juxtaposed with a gap or space separating them and *no* apparent *phonetic change*. The case that they stand for $n \times 10^k$, in other words, that *multiplicative compound numbers need not necessarily involve grammatical composition*, finds support from the fact that each name in such a number is declined in accordance with the appropriate gender, number and case. That may not remove all doubt however and we shall take a look at a few apparently ambiguous cases in the next section.

The words *daśa*, *śata* and *sahasra* appear frequently either individually to denote 10, 100 and 1,000 or as parts of the names of compound numbers. The former use, again, need not detain us as the names are in principle arbitrary (they may not even have any meaning) and no rules of any sort are involved.[24] Sometimes such words, in particular *sahasra*, are employed in the sense of numerous or perhaps even innumerable. A possible example is *daśa ... sahasrāṇi* (1.53.6)[25]which Sāyaṇa says signifies unlimited or unbounded (*aparimitāni*). The grammatical reason for such an interpretation is not obvious – it is consistent with the meaning 'powerful' – though the contexts in which similar usages occur may often provide a clue. A clear-cut case is (10.90.1) speaking of Puruṣa having a thousand heads, a thousand eyes and a thousand feet that makes no realistic sense unless *sahasra* is taken to mean innumerable. Another possible instance is (4.32.18),

> *sahasrā te śatā vayaṃ gavāmā cyāvayāmasi*
>
> *asmatrā rādha etu te*

which Griffith's translation of the *Ṛgveda* ([Gr]) renders as "We make a hundred of thy kine, yea, and a thousand, hasten nigh: So let thy bounty come to us." Such usages have their exact modern parallel in expressions like 'hundreds and thousands'. A more intriguing example occurs in (4.26.7), "... *sahasraṃ savā ayutaṃ ca sākaṃ*..." which, because of the conjunctions *ca* and *sākam*, we will normally render as referring to '1,000 and 10,000' = 11,000. This agrees with O'Flaherty's translation ([DO'F]), "(he brought it for) a thousand and ten thousand (pressings at once)". But Sāyaṇa says that it should be taken to mean *aparimitasaṃkhyākam*. Such uncertainties, it has to be stressed, are not fatal to the number naming paradigm that our enquiry is concerned with. If Sāyaṇa's authority is not to be questioned, the phrase is simply to be discarded from our databank of number names and if it really is the name of a number, that number cannot but be 11,000.Indeed, in the early phase of the evolution of number names that the *Ṛgveda* represents, it appears to us to be obligatory to look at every compound number name, in particular apparently conjoined powers of 10 (another example is *sahsriṇaṃ ca śatinaṃ ca*, 1.124.13), with an open mind: does it really represent a specific number or something more qualitative like 'lots and lots' ?

Even if the *aparimita* interpretation of such words and phrases is accepted, the numerical significance of *ayuta* itself is without doubt 10,000 as it remained throughout the long history of Indian numeration (see Hayashi's lists in [Ha]). The word occurs in isolation in, for instance, (8.1.5) (*na sahasrāya nāyutāya vajrivo na śatāya*) and also in multiples, e.g., *catvāryayutā* = 40,000 in (8.2.41). But the more common name for 10,000 is *daśa ... sahasrā* (8.46.22) or *sahasrā daśa* (8.5.37, 8.6.47) (this last is an example of multiplicative compound numbers not involving grammatical compounds) or its recognisable variants like *daśabhiḥ sahasraiḥ* (8.1.33).[26] Once again, the presence or absence of grammatical composition in multiplicative compound numbers can perhaps be put down to continuing experimentation.

There are rare occurrences of the words *niyuta* and *arbuda* which, soon afterwards (in the *Taittirīya Saṃhitā*), denoted numbers (10^5 and 10^7). But, according to Sāyaṇa, in the *Ṛgveda* they are not names of numbers but of objects or beings such as demons.[27]

We note in passing that *maṇḍala* 8 is exceptionally abundant in the names of powers of 10 in various combinations. One verse in particular (8.46.22) almost flaunts them:[28]

> *ṣaṣṭiṃ* **sahasrā***śvasyāyutāsanamuṣṭrānāṃ viṃśatiṃ* **śatā daśa**
> *śyāvīnāṃ* **śatā daśa** *tryaruṣīṇāṃ* **daśa** *gavāṃ* **sahasrā.**

Logically, we should be dealing next with multiplicative compounding rules in as much generality as possible, i.e., those which prescribe the names of numbers of the form n × 10^k: 20, 30, ...; 200, 300, ...; 2,000, 3,000, ...; and 20,000, 30,000, ...; the next and final step being that of adding up such multiples. Such a linear path is difficult to follow in practice because it is relatively rarely that the names of multiples occur in isolation and so we are obliged to unravel the additive compounding rule as well to get at them. More seriously, multiplicative compounding is less transparent and systematic than the additive; indeed, as will be seen presently, one way in which certain multiplicative names are identified is by elimination, i.e., by determining that the compound in question cannot be additive.

The names of multiples of 10, in increasing sequence and ending with 100, figure in two consecutive verses of *Ṛgveda* (2.18.5 and 2.18.6); they supply us with as valuable a list as the famous lists of powers of 10 in the *Taittirīya Saṃhitā*. The passage is

> *ā viṃśatyā triṃśatā yāhyarvāṅā catvāriṃśatā haribhiryujānaḥ*
>
> *ā pañcāśatā surathebhirindrā"ṣaṣṭya saptatyā somapeyam*
>
> *āśītyā navatyā yāhyarvāṅā śatena .....*

Thus, without any doubt, the names *viṃśati*, *triṃśat*, *catvāriṃśat*, *pañcāśat*, *ṣaṣṭi*, *saptati*, *aśīti*, *navati* stand for 20, ..., 90. The reason we have written out the list in full is that their derivation from the name of 10 and the names of the (atomic) multiplers does not seem to follow one uniform rule. According to Monier-Williams ([M-W]) the etymology for *viṃśati* is as follows. The word *daśa* (stem: *daśan*) is the name for 10 and the combination of *daśati* with *dvi*, followed by certain phonetic transformations, gives *dvi* + *daśati* → *viṃśati*. Clearly, similar derivations can be given for *triṃśat*, *catvāriṃśat* and *pañcāśat*. The phonetic transformations required in the names of the other multiples are more drastic: it is not so easy to track the changes in, for example, *aṣṭa* + *daśati* → *aśīti*. Ultimately all we can say is that there are traces of the composing elements in the final name; the true guarantee of the interpretation is that they are long-established and so, by continuity of tradition, beyond questioning. Indeed, the *vṛtti* to Pāṇini 5.1.56, which itself is a list of multiples of 10 uncannily reminiscent of the list in *Ṛgveda* cited above:

paṇktiviṃśatitriṃśaccatvāriṃśatpañcāśat-

ṣaṣṭisaptatyaśītinavatiśatam

says that the names of multiples of ten are anomalous: *ruḍhiśabdāḥ nipātyante* ([Va]).

Examples of names of multiples of 100 by themselves are 200 (*dve ... śate*, 7.18.22) and 500 (*pañca śata*, 10.93.14). Apart from these, 300 occurs in several places as part of larger numbers, e.g., *trīṇi śatā trī sahasrāṇi ... triṃśacca ... nava ca* (3.9.9). A legitimate question here is how sure one can be that the number preceding *śata* or *sahasra* is a multiplier and not meant to be added. A first answer is that reading them additively would lead to an unnatural way of expressing numbers: why would one invoke 1,145 – the context of the poem provides no special reason – as 103 + 1,003 + 30 + 9 instead of the decimally canonical 300 + 3,000 + 30 + 9? Confirming the correctness of this answer requires grammatical considerations of the kind we illustrate in section 7 below. Traditional readings support the interpretations of *dve ... śate* and *pañca śatā* as 200 and 500 and of *trīsahasrā* as 3,000.

Aside from *trīsahasrā*, there occur a number of other atomic multiples of 1,000: we mention 4,000 (*catvāri ... sahasrā*, 5.30.14 and *catuḥ sahasram*, 5.30.15) and 8,000 (*aṣṭā ... sahasrā*, 8.2.41). There is also the curious case of 6,000 being expressed in two different ways, as 6 × 1,000 (*ṣaṭ sahasrā*) and as 60 × 100 (*ṣaṣṭiḥ śatā*) in the same verse 7.18.14 whose interpretation no two commentators seem to agree on.

When it comes to multiples of 10,000, there is a departure from the pattern followed for 100 and 1,000. Just as 10,000 itself is more often named as 10 × 1,000, so a multiple of 10,000 is almost always represented as the multiplication of 1,000 by the corresponding multiple of 10. There are quite a few examples of this construction: 30,000 (*sahasrā triṃśataṃ*, 4.30.21), most frequently 60,000 (*ṣaṣṭiḥ sahasram*, 1.126.3; *ṣaṣṭiṃ sahasrā*, 6.26.6, 8.4.20 and 9.97.53) and 90,000 (*navatiṃ sahasrā*, 10.98.11); there is also the interesting 99,000 (*navatirnava ... sahasrā*, 10.98.10) which, as it happens, is the largest number named in the *Ṛgveda*. Finally, in line with the relative rarity of the name *ayuta*, we have one instance in which 40,000 is called *catvāryayutā* (8.2.41).

The multiplicative interpretation of the names involving 1,000 and 10,000 cited above are consistent, with very little uncertainty, with grammatical analysis as we shall illustrate in a few examples in the next section.

7. Number Names in *Ṛgveda* – Resolving Ambiguities

The only two functional objectives of numerical composition in *Ṛgveda* are addition and multiplication. (Subtractive number names appear first, sporadically, in *Taittirīya Saṃhitā* where the process is explicitly indicated, e.g., *ekānnaviṃśati*, *ekānnacatvāriṃśat*, etc.). This fact is of great help since ruling out a given compound number as additive establishes it with certainty as multiplicative in those cases where a direct grammatical analysis does not lead unerringly to the identification of multiplicative composition.[29] In order to be able to say that such a number is multiplicative, it is therefore necessary (and, in principle, sufficient) to deal first with the rules for additive composition, a relatively easier task.

In the formative phase that *Ṛgveda* represents, the additive rules are, generally, explicitly descriptive of the arithmetical process at work. The reason perhaps is that the grammatical equivalent of addition, conjunction, was routinely and widely implemented throughout *Ṛgveda* in non-numerical contexts through connectives like *ca*, *sākam*, etc. Of the numerical instances, we give only a few out of very many: *śatamekaṃ ca*, 100 + 1 (1.117.18); *triṃśataṃ trīṃśca*, 30 + 3 (3.6.9); *saptatiṃ ca sapta ca*, 70 + 7 (10.93.15); *śataṃ ... sapta ca*, 100 + 7 (10.97.1); *navabhiḥ ... navatī ca*, 9 + 90 (10.39.10). Then we have the frequently cited verses from 1.164 about the number of paired days and nights in a year: *sapta śatāni viṃśatiśca*, 700 + 20 (1.164.11) and *triśatā na ... ṣaṣṭirna*, 300 + 60 (1.164.48).

But the expressed use of conjunctions for addition, leading eventually to the formation of *dvandva* compounds, was not a universal practice. The *dvandva samāsa* always reflects the additive rule in operation but, according to the traditional authorities, the converse is not necessarily true. This is most clearly brought out in the case of the smallest, and hence presumably the earliest formed, compound numbers 11 to 19. Words like *ekādaśa, dvādaśa, ..., navadaśa* can be analysed as *dvandva* compounds (e.g., *dvādaśa as dvau ca daśa ca*, 2 and 10) or, more interestingly since it brings in the idea of succession, as a subclass of *tatpuruṣa* (*dvyadhika daśa*, 2 more than 10) (*vṛtti* on Pāṇini *sūtra* 6.3.47 [Va]). Similar considerations apply to the names of all numbers between 10 and 20 and indeed to most numbers of the form of a multiple of 10 plus an atomic number, e.g., *trayastriṃśatamā* (3 + 30, 1.45.2), *catustriṃśad* (4 + 30, 1.162.18).[30] Whichever way they are analysed and whatever *samāsa* they are assigned to, we can reasonably take such names as marking the evolution of the identification of a number from a description of its arithmetical content to a proper number name through the operation of *sandhi* transformations.

The nonatomic number that is found most often in the *Ṛgveda* – in supposedly early and late *maṇḍalas* alike – is 99 and the variations in its nomenclature allow a fascinating glimpse into how the compounding rules intersect with the need to express 90 + 9 as a precisely defined number. Here is a sample: *navatīrnava* (1.84.13), *navatirnava* (4.48.4, 10.98.10), *navānāṃ. navatīnām* (1.91.13), *nava ... navatiṃ ca* (2.14.4), *navatiṃ ca nava* (2.19.6), *nava sākaṃ navatiḥ* (4.26.3), *navabhiḥ....navatī ca* (10.39.10). Not only do we have here the arithmetical recipe in several variants leading finally, in *navatirnava*, to a proper number name in which the connectives have been transformed away, but also a grammatical nod to the commutativity of addition: 90 and 9 are conjoined in either order.

Additive composition in all these examples either involves the use of conjunctions or leads to compound words which can be analysed as additive with a degree of certitude. Do these two general principles cover all additive compounds? In other words, are all other compound numbers multiplicative? For a decisive answer, we have to subject number names not falling into these two categories to an independent grammatical analysis and, in most cases, this is possible with help from Pāṇini, supplemented by the padapāṭha reading of the text. A name-by-name analysis being obviously beyond the scope of this article, we confine ourselves here to a small selection of putative multiplicative compounds (including cases in which neither factor is a power of 10), and not in exhaustive detail. Apart from one or two instances of ambiguity, the conclusion will be that they are indeed multiplicative.

Generally speaking, most numbers in the *Ṛgveda* function as adjectives: so many of something. In compound numbers in which no *samāsa* operates between the two numbers composed, the first is functionally an adverb qualifying the second which continues to function as an adjective. The rule *dvitricaturbhyaḥ suc* (Pāṇini *sūtra* 5.4.18) covers this case and requires that the *taddhita* suffix *suc* (= s which changes to the *visargaḥ*) be added to the first number when it is 2, 3 or 4. The purpose of *suc* is to keep track of the repetition of an action, in this case that of counting as determined by the second number and the whole is characterised as *āvṛttivācaka*. The result of the repetition is the multiplication of one number by the other.[31] Thus *triḥ sapta* is 7 counted thrice = 3 × 7 (*triḥ sapta* or *trir*(*asmai*) *sapta* occurs quite often, e.g., 9.70.1) and *dvirdaśa* similarly is 2×10 (1.53.9). The latter is therefore an alternative name for *vimśati* in which *dviḥ* is a numerical adverb and which is analysed as *dvirāvṛttāḥ daśa*. (The other way of combining 2 and 10, namely *dvādaśa*, definitely connotes 12 as, according to Pāṇini (*sūtra* 6.3.47), *dvi* changes to *dvā* in compounds other than *bahuvrīhi*). Similar justifications hold, apart from 3×7, for 2×5 as *dviryaṃ pañca* (4.6.8 and 9.98.6), 3×11 as *trīrekādaśam*, etc., and should hold in fact for any product with 2, 3 or 4 as the prefactor.

What is interesting in these examples is that they cannot naturally be analysed as *dvandva* (*m* and *n*) or as some variant of *tatpuruṣa* (*m* more than *n*), i.e., as the additive rule at work.This principle of elimination can be extended: whenever a compound number name does not have an additive structure when analysed according to established grammatical principles, it must be considered to be multiplicative even if the first factor is not 2, 3 or 4. The reader will have noticed that we have already made use of this line of argument, sometimes tacitly, in discussing multiples of powers of 10.

Having said this, we hasten to note that *triḥ ṣaṣṭih* as it occurs in (8.96.8) is not, unlike *triḥ sapta*, multiplicative. It is in fact additive, as we know from Sāyaṇa who knows from tradition that the verse refers to the 63 (9 groups of 7 each) *maruts*: *ṣaṣṭitryuttarasaṃkhyākāmarutaḥ*. There is also grammatical authority for this. By virtue of the rule *supām sulukpūrvasavarṇā* . . . (Pāṇini 7.1.39), *triḥ* here (nominative singular, *su* affix) is a replacement for *trayaḥ* (nominative plural, *jas* affix) and does not count the repetition of an action - the *taddhita* rule is suspended.[32]

It remains for us to observe, given the importance of grammar as the final arbiter of the choice between the additive and the multiplicative interpretation of a given compound name, that the last word in this choice must lie with an authority earlier than Pāṇini, namely the word-by-word reading of the text, the *padapātha*.[33] A sampling of putative multiplicative names does confirm that they are in fact multiplicative but not always without ambiguity. For illustration, consider four apparently similar names that we have already met: *catvāryayutā* (8.2.41), *catvāri sahasrā* (5.30.14), *catuḥ sahasraṃ*. (5.30.15) and *catuḥ śataṃ* (8.55.3). The first is a *sandhi* in which both words are in accord as regards case, gender and number (nominative neuter plural), making the case that *catvāri* as adjective (or adverb) qualifies *ayutā* as noun (or adjective). The resulting inference that *catvāryayutā* means 40,000 is, further, supported by the context of the verse in which it occurs.

Now, in the *padapaṭha*, *catvāryayutā* and *catvāri sahasrā* have identical separations and accents, letting us conclude in turn that the latter is 4,000.

The situation is much less clear for *catuḥ sahasraṃ and catuḥ śataṃ*. The *padapaṭha* assigns the same structure to both phrases, *udātta* accent on *ca* and *avagraha* between the two words. The *udātta* would make them *dvandva* compounds (no *suc* repetition): they should mean 1,004 and 104. But the *avagraha* normally negates the *dvandva* reading. So what is one to make of them?

Let us note also that there is at least one place where Sāyaṇa confesses uncertainty: his explanation for *viṃśatiṃ śatā* (8.46.31) is "*viṃśatiṃ ca śatā śatāni ca athavā śatānāṃ viṃśatiṃacikradat*", either it is 20 + 100 or it is 20 × 100. And, finally, we have the ambiguous verse (7.18.14) which has been variously read:

ni gavyavo'navo druhyavaśca ṣaṣṭiḥ śatā suṣupuḥ ṣaṭ sahasrā

ṣaṣṭirvīrāso adhi ṣaḍ duvoyu viśvedindrasya vīryā kṛtāni

The difficulty here is not primarily grammatical but the fact that both *ṣaṣṭiḥ śatā* (a perfect example of *suc* in operation even when the multiplier is not 2, 3 or 4) and *ṣaṭ sahasrā* mean 6,000. It is not impossible that there is only one number in the verse, 6,000 + 6,000 + 60 + 6, but that would appear to be an unnatural way of expressing 12,066. And there is no conjunction connecting the two 6,000s to one another or to the 66. What then do the numbers 6000, 6000 and 66 refer to?[34]

In the end, we should not forget that difficulties later commentators (and of course modern readers) come up against in analysing a given number name do not imply that the poets themselves were uncertain about its meaning – the occasional deviation from grammatical and even metrical norms is not unknown in the vedic corpus. In any case, the few examples of uncertain interpretation cited do not alter the fact that the vast majority of number names in *Ṛgveda* can be given a unique numerical sense. Our aim has been to subject these names to whatever method of study will lead to assigning to each of them that unique number. Grammatical analysis is the most powerful such method, but our forays into grammar are meant primarily only to resolve the residual ambiguities that are bound to arise in the interpretation of number names in a language-in-making such as vedic Sanskrit. What is remarkable is that, at the end of the exercise, the evidence in the *Ṛgveda* for the mastery of decimal enumeration is as solid as we have found it to be. To conclude, how can it be doubted that *ṣaṣṭiṃ sahasrā navatiṃnava* (1.53.9) is 60,099 and that *trīṇi śatā trī sahasrāṇi... triṃśacca ... nava* (3.9.9 and 10.52.6, in identical words) is 3,339?

8. Beyond Counting

As is well-known, the mastery of decimal enumeration gave rise in time to a fascination with very large numbers, expressed chiefly in the compiling of long lists of powers of 10 (for such lists from different sources and periods, see [Ha]), among the earliest being passages from *Taittirīya Saṃhitā* one of which we have already cited. There are in fact good indications that the principle of generating ever larger numbers by taking higher and higher powers of 10 was well understood relatively early. *Taittirīya Upaniṣad*, later than the eponymous *saṃhitā* but much earlier (pre-Buddha according to Staal [St6]) than all the other lists as catalogued by Hayashi, has a poem (2.8) about the bliss of *brahman*. It is too long to quote here but begins by defining the bliss of a young man accomplished in every way as one unit, then goes on to a sequence of increasingly divine beings, each a hundredfold as happy as his predecessor until, in ten stages, it gets to *brahman*.

There are no names given to any of the powers of 100 but 10^{20} is a very large number indeed even by later standards, higher than the 10^{17} which the author of *Yuktibhāṣā* used more than two thousand years later as a working substitute, so designated, for infinity. It does not say, unlike *Yuktibhāṣā*, that one can go on – the bliss of *brahman* is great but, technically, not infinite. There is also other early evidence for a sharper awareness of the unboundedness of numbers, not always tied to (exponentially) increasing powers of 10. *Taittirīya Saṃhitā* (7.2), again, counts sacrificial oblations in many different linear sequences, starting with a low number (1,2,3 etc.), skipping 2,3,4,5 etc. and going up generally to 100 (with breaks in the middle) but with a coda, *sarvasmai svāhā*,[35] attached. We are not concerned with the notion of infinity in this article and so will leave aside the subject of very large numbers.

Accurate enumeration leads naturally to its use in arithmetic. We have noted more than once that an intuitive understanding of basic arithmetical operations is a prerequisite for decimal counting. What the latter does in turn is to act as the formal foundation for the transformation of the arithmetic of atomic numbers, empirically done and memorised, into a fully rule-based science, algorithms applicable to numbers as large as we please. But beyond the needs of decimal enumeration itself – i.e., multiplication in which at least one factor is a power of 10 and addition of multiples of different powers of 10 – the evidence from the *Ṛgveda* for general addition and multiplication is sketchy. For addition the only example we have found is *pañcadaśa sākaṃ...viṃśatiṃ* : 35 = 15 + 20 (*RV* 10.86.14). Since the use of *ca* and *sākaṃ* in the formation of additive number names was common, the rarity of general addition probably has more to do with absence of poetic necessity than with ignorance of the concept; the verses (*RV* 1.164.11) and (*RV* 1.164.48) about the number of days and nights in a year are implicit proof that 360 + 360 was known to be 720 (or that 2 × 360 = 720).

There are a few instances of general binary multiplication some of which we have already cited in section 7. The others are equally elementary, involving small numbers or their tenfold multiples. They are all of the type that belong to a memorised multiplication table, not requiring 'long multiplication', i.e., algorithms depending critically on the place-value notation for their computation. It serves little purpose to list them. On the other hand there is an interesting and unique example of three numbers being multiplied together: (*usrās*) *triḥ sapta saptatīnāṃ* = 3 × 7 × 70 (*RV* 8.46.26). Apart from the intrinsic arithmetical interest, it also illustrates well the multiplicative compounding rules discussed in section 7. As seen there, *triḥ sapta* is 3 × 7 with *triḥ* as a numerical adverb. The word *saptatīnāṃ* which is feminine plural in the genitive case makes the meaning clear: 21 of 70; in other words 3 × 7 functions as an adjective qualifying 70.

The first post-Ṛgvedic description of an arithmetical process that we have seems to be a passage from the *Śatapatha Brāhmaṇa* (10.4.2) (probably a century or two before the earliest *Śulbasūtra*) about the division of Prajāpati's body (identified at one level with the year and at another with the *vedi*) into 720 parts (days and nights or bricks making up the *vedi* respectively).[36] It is the earliest reference there is to the process of division. And, in the distinction drawn between exact divisors and numbers which are not, we see the operations of division with and without remainder, in other words the foundation of decimal enumeration, overtly at work. Very soon thereafter comes the more elaborate arithmetic involving fractions of the earliest *Śulbasūtra*, those of Baudhāyana and Āpastamba (800-700 BC). This and the subsequent development of arithmetic, both practical and formal, until the time of *Yuktibhāṣa*, will be treated elsewhere.

9. Numbers and Grammar

We may summarise our main conclusions in two points.

1) An oral expression of a general, abstract place-value principle is necessarily realised by a systematic, rule-based method of number

nomenclature. Such a nominal realisation is as precise and unambiguous as the written one for the purpose of enumeration. The formulation of the place-value principle, no matter how it is realised, is itself impossible without an implicit grasp of the place-value algorithm involving division with remainder.

2) This theoretical picture has very strong empirical support from the decimal number nomenclature in the *Ṛgveda*. Far from being limited to cardinality-matching with the occasional number name thrown in, the four inputs required to implement the ideal naming system, i) names for the atomic numbers, ii) names for powers of 10, iii) a grammatical rule(s) for naming multiples of powers of 10 by the atomic numbers and iv) another rule(s) for adding up the multiples of various powers of 10 (including the 0th power, i.e, the atomic numbers themselves), are all on full display in all the *maṇḍalas* of *Ṛgveda*. The great majority of the variations in the application of the rules are those required to meet grammatical or metrical exigencies and are not a source of ambiguity.

Evidently, then, the systematics of the rules governing number names in the *Ṛgveda* makes a very strong case for the oral expression of the place-value principle to be placed on par with the symbolic written expression, with the names of the atomic numbers playing the same role as the numeral symbols and the names of powers of 10 that of 'position'.

With regard to point 1), it is not our intention to suggest that the number nomenclature of the *Ṛgveda* arose out of a conscious appreciation of the place-value principle. Abstraction for its own sake, without a practical context, was never much in favour in India. But that did not prevent the setting down of rigorous rules in particular fields of study that called for a rule-bound approach. The outstanding instance of this is of course the study of language, most famously exemplified by Pāṇini and the laws of Sanskrit grammar. They are claimed to have been arrived at as a description of the linguistic usages of his time and were thus a codification of practices from earlier times. While the rules devised by Pāṇini to universalise the applicability of grammatical constructions have no direct recorded antecedents, their roots in the vedic tools of text analysis, as embodied in the *padapāṭha* for instance, seem to be well grounded (see for example [St3]). More importantly, even without the benefit of analytical tools, it is an obvious fact that much of vedic grammatical usage is in consonance with post-facto Pāṇinian principles. If that were not so we would not have been able, in the very limited domain of our interest here, to fit the number naming rules into the framework of the classification of word-composition which was only put in place later. A reading of any modern textbook dealing with vedic compound words leaves one with the impression that this classification is essentially taxonomic – Pāṇini's attention to exceptions to normative rules (and exceptions to exceptions) already bears witness to this – the rules generally a step behind the practice. Macdonell for instance is careful to say that such-and-such vedic construction corresponds to what later grammarians designated as such-and-such *samāsa*.[37]

In so far as number names are concerned, our examination of *Ṛgveda* strongly supports this scenario: we hear with our ears as it were the as yet uncodified rules governing word-compounding at work. As we have noted, the structural issues connected with the creation of a system of number names are few in number and enormously simpler than the full-edged grammar of a natural language that subsumes them; perhaps that is why the need for explicitly enunciating the principles underlying the decimal number system was never felt once Pāṇini had dealt with grammar as a whole (until we get to Kerala in the 15th century, where new questions about the "nature of numbers" were asked and answered).

Just as it is impossible to think that an abstract place-value principle preceded the invention of decimal numbers, so it is impossible to think of their names in *Ṛgveda* as being inspired by Pāṇinian rules of nominal compounding ([Ka1]).[38] We should rather view them as part of the vast storehouse of pre-existing grammatical expressions which linguists like Pāṇini subsequently drew upon. Apart from radically transforming our understanding of the genesis and early evolution of decimal numbers, mainly of interest to historians, for grammarians they provide a relatively easily analysed set of linguistic objects to work with.

The relative neglect of numbers in grammatical studies is easy to understand: number names are not the first objects to come to mind in the study of subtle questions of grammar. And, with one exception ([Mur]),[39] no one seems to have explicitly recorded that the *Ṛgveda* is so incredibly rich in number names, upwards of two thousand five hundred in all, fortyfive in one single poem (*RV* 1.164). It is as if the poets and the bards were exhilarated by this heady new discovery, one which put them on par with Agni.

Following Thibaut's pioneering work on the *Śulbasūtra* in the 1870s and his view (entirely correct at that time) that these texts contain "the earliest geometrical and mathematical investigations among the Indians" ([Th]), it has become routine to say in scholarly circles that the geometry of the *Śulbasūtra* is the first manifestation of the mathematical heritage of India. But these very texts testify to a mature arithmetical culture as well. While little is known with confidence about the antecedents of the geometry, there can be no doubt that the arithmetic would have been impossible without the control over numbers afforded by decimal enumeration. Even though there is no self-contained early work devoted to the subject, we would make a case for considering the circle of concepts that contributed to the invention and mastery of decimally organised numbers, at a substantially earlier time, as part of the first flowering of an exact mathematical science in India; our analysis of number names in the *Ṛgveda* provides enough evidence in support of this suggestion. That such a development in an orally literate environment should bring in aspects of grammar related to the creation of new names from old is then an inescapable consequence – number names are the first manifestation of the Sanskrit of science. And that takes us once again to the Bhartṛhari passage quoted earlier about building new verbal expressions from existing ones as new numbers are built from 'first' numbers. Indeed, this passage (*Vākyapadīya* 1.87) is preceded by others (1.82-83 for instance) expressing the idea that while the apprehension of a linguistic construct is impossible without apprehending its component parts, it is not fully determined by them, exactly as in the meanings of numbers. We are compelled to wonder whether vedic literacy and numeracy did not evolve in tandem and whether the common ancestry of numbers and grammar in the *Ṛgveda* is not at the root of the ties – most notably their recursive structure – that bound them over such a long period in India.

Acknowledgements

This work has benefited from the special knowledge of many, of whom we would particularly like to thank Chandrashekhar Cowsik, Kavita Gangal, Madhavi Narsalay and Ranjani Ramachandran. We are grateful to the late Frits Staal for many useful suggestions on an early version of the paper, Uma Vaidya for valuable advice on some questions of grammar and two anonymous referees for several constructive remarks. Above all, our thanks are due to Kamaleswar Bhattacharya and Pierre-Sylvain Filliozat for their interest in this work over a long period and their generous help and detailed comments, enabling us to bring this project to a close.

This study was initiated during the tenure of a Senior Award to P.P.D. of the Homi Bhabha Fellowships Council at IUCAA, Pune. B.B. received early support from the Jamsetjee Tata Trust. The hospitality of IUCAA to B.B. and UM-DAE Centre for Excellence in Basic Sciences, Mumbai to both of us is acknowledged gratefully.

Notes and References

1. In Indian mathematical literature, this principle is referred to as one of positional variation (*sthānabheda* or a synonymous phrase). The term 'positional' signifying relative location or order in linear space is too readily associated with a written notation or, as we shall specify later, a written representation of a general, abstract, place-value principle; 'place-value' is our preferred term when we speak of a variety of expressions of that principle rather than just the written one. By 'number' without qualification we shall mean natural or counting numbers, positive or nonnegative integers, depending on the context.

2. It is an amusing sidelight that, as late as the mid-1850s, some influential British mathematicians felt that the use of negative numbers in arithmetic lacked a proper foundation, see [Mum].

3. It seems correct to say that this cautious attitude to epistemic issues was not confined to a particular religion, sect or philosophical school, as has been widely discussed in the literature. A very detailed account can be found, among numerous others, in Matilal's book [Mat].

4. We can glimpse a more abstract approach beginning to emerge, under the impetus of the new demands of the work of Mādhava, the founder of the Nila (Kerala) school of mathematics and astronomy, in the late 14th – early 15th c. AD. That and related issues will be taken up elsewhere.

5. Particularly relevant is the work of the Nila school, especially the philosophical sidelights in the work of Nīlakaṇṭha. See for instance [Na].

6. For any choice of number b for the base, we use this term to denote the numbers 1; 2; 3; …; b - 1. It is preferable to other commonly used words like digit (too closely linked to 10 as the base) or numeral (may support other interpretations). The terminology conforms to Indian practice at least from the 1st century onwards (see section 5).

7. Ambiguities inherent in the analysis of nominal compounds continue to interest scholars; a recent example is [Gi]. There are also a number of well-known textbooks which can be consulted for the more elementary aspects, for instance Macdonell's books on vedic grammar [Mac2, Mac3].

8. In particular, we have nothing to say about the contextual or symbolic significance of specific numbers, a topic on which famous scholars, for instance Gonda, have written in the past ([Go1], [Go2]). A recent article by Murthy ([Mur]) has a list of numbers from the *Ṛgveda*, a subset, by no means exhaustive, of the numbers actually occurring there, which are said to have symbolic significance. We are also not primarily concerned with other approaches to the connection of numbers to grammar such as those relying heavily on linguistic or computational theory. (Kadvany ([Ka1]) provides a starting point for the retracing of this line of inquiry).

9. See, apart from articles (both learned and popular) too numerous to cite individually, the books of Datta and Singh ([DS]) and Plofker ([Pl]).

10. The Sanskrit expression equivalent to and translating as 'decimal' is *daśamāna*, 'measurement by ten'.

11. As described operationally above, based numbers may appear to have more structure than numbers as defined by Peano's axioms which formalise the idea of succession, without the need to choose a base. It is easy to see that that is not the case – the place-value construction conforms strictly to the axioms. The early 16th C. Malayalam text *Yuktibhāṣā* ([TA], [Sa], chapter 1) actually takes the first step in linking decimal numbers to the notion of succession, very likely on account of its use of inductive proof methods.

12. The vision/illumination ambiguity is not unique to Agni. Much later, in Mahāyāna Buddhism, Amitābha ("immeasurable radiance"), the transcendental and cosmic version of the Buddha, is endowed with the gift of being able to see all and the name of his most revered and popular Bodhisatva emanation, Avalokiteśvara, means literally "the lord who looks down (on all)". Perhaps it is only a coincidence but the philosophy and worship of Amitābha became greatly popular in an epoch (around the beginning of the common era) when preoccupation with very high powers of 10 reached a climax and cosmogonic speculation about (infinitely) multiple universes in time and space pervaded the Mahāyāna world-view; Amitābha was the presiding deity of the totality of these multiple worlds.

13. Musicians we have talked to are unanimous that they are oblivious of any conscious effort at keeping to the *tāla* as they perform.

14. It is worth emphasising that the gestures used in counting are not to be thought of as what are called *mudras* ([St4]). In the recitation of vedas these latter are formalised gestures whose function is to visually represent and accompany the sounds of the chant, not to count how many repetitions there are. In contrast, the moving of the thumb over the finger joints that we are concerned with here has no symbolic

significance. Its only purpose is the practical one of keeping count; the vocal organs being occupied in chanting, it is hardly possible for them to pronounce the number names at the same time

15. This is the distinction we have sought to convey from the beginning by the phrase 'atomic numerals' or 'atomic numbers'. We might add that the use of the word 'atom' in this context has the sanction of Bhartṛhari who speaks of the atoms (*aṇu* and *paramāṇu*) of word or speech (*śabda*) gathering together, by the manifestation of their own capacity, like clouds (*Vākyapadīya*, 1.110-111).

16. The authorship of the *vṛtti* is not authenticated and has generated controversy. It is sufficient for us that it reflects the views of Bhartṛhari, whether the commentator is himself or an immediate disciple.

17. The translation is ours, aided by Biardeau ([Bi]) and Iyer ([Iy]), and is not absolutely word-for-word.

18. The decimal place-value principle served as a paradigm not only for grammarians. Staal ([St7]), following Ruegg (to whose article we have no access), also cites a passage from the Buddhist philosopher Vasumitra (1st-2nd century AD) to the effect that a dharma, though its 'substance' is the same, has different significances in different 'states' (*avasthā*), "like a marker or counter in reckoning which in the unit position has the value of a unit, in the hundred's position has the value of a hundred and in the thousand's position has the value of a thousand". This is very reminiscent of the later flower-in-the-garland analogy (see footnote 21 below). A thorough search will surely reveal other instances of decimal place-value numbers serving as a model in varied domains of enquiry.

19. An English translation of the work by K.V. Sarma has recently been published ([Sa]). There is also the admirably annotated critical edition (of the first part) in Malayalam by Rama Varma Tampuran and Akhilesvara Ayyar ([TA]).

20. For a differently worded rendering, see Sarma ([Sa]), section 1.2.

21. Tampuran and Ayyar ([TA]) supply an illuminating footnote at this point on the relationship of *vikṛti* to its original *prakṛti* along with the illustrative example of a flower in a garland which, though it partakes of 'flowerness', serves a purpose distinct from that of an individual flower. The similarity with the idea of *prakṛti* numbers serving as the constitutive elements of the string ("garland") that is a *vikṛti* number is obvious.

22. Sarma's translation ([Sa]) of the relevant sentence is "... there will be no limit to the way numbers are designated and it would be impossible to recognise the numbers ..." which is not quite what the original says: the replacement of the (non-Sanskrit) Malayalam noun for 'name' by the verb 'designate' and 'know' by 'recognise' appears to be unjustified, as is that of the causal 'hence' by the conjunction 'and' (the text clearly conveys the causal relationship between naming and knowing).

23. Nearly identical names for the odd atomic numbers 1 to 9 occur on a clay tablet as part of the Mitanni treatise on horses and chariots by Kikkuli, dated to the 14th century BC, well before the compilation of the poems in *Ṛgveda* though not perhaps before their composition. The Mitanni country was not so far from the part of Mesopotamia where the tablets with symbolic positional numbers from an earlier period were found; so it is a surprise that the numbers are actually written as their names, in words. A case of an oral tradition coming across a script in which to express itself?

24. The name *sahasra* may present an interesting exception. It is plausible that it is formed from *sahas* meaning 'power' and *ra* expressing possession in which case *sahasra* will signify 'powerful'. We thank Professor Pierre-Sylvain Filliozat for this observation. Could it be that the name came to be given to the largest power of 10 then in current use and that, given the relative rarity of the next power 10,000 (and the absence of all higher powers) and the greater frequency of *daśa sahasra* as compared to *ayuta* in *Ṛgveda*, the step up from 1,000 to 10,000 was not made automatically as it were? Professor Filliozat also points out that *daśa* and *śata* can be considered to be 'atomic' in the grammatical sense, i.e., that they are not compound words. Our use of the qualifier 'atomic' is restricted to the numerical sense, i.e., those numbers which are not formed from smaller numbers, the *ādyasṃkhyā* of Bhartṛhari.

25. In this section and the next, all references to verses from the *Ṛgveda* are given without the prefix *RV*.

26. There is at least one instance of 1,000 being called *daśa śatā* (5.62.1).

27. In post-vedic literature the names of high powers of 10 are generally words which have other non-numerical meanings, e.g., *padma*, *samudra*, etc. Many such words already occur in the *Ṛgveda* in their literary sense.

28. In the context of the verse, *ayuta* here is very likely not the name of a number, but the negative of the past participle of *yu*, thereby signifying 'unyoked', 'unbound' etc. In the spirit of the meaning and significance of *sahasra* (see footnote 24), it would seem to be an appropriate name for the largest power of 10 in *Ṛgveda* – compare Sāyaṇa's *aparimita* interpretation of *ayuta* above. The use of names such as these having certain literal meanings does not alter the fact they are, as names of numbers, arbitrary.

29. This problem has a history. Colebrooke in his well-known translations from Brahmagupta and Bhāskara II ([Co], p. xxxvii) as well as Burgess ([Bu]), probably following him, took references to Āryāṣṭāśata to mean that (the three main chapters of) *Āryabhaṭīya* has 800 instead of 108 verses (before, of course, Kern's edition was published).

30. A list of such number names in the vedas can be found in [Mac2].

31. Arithmetical terminology in general is a faithful indicator of the grammatical considerations that always guided it. In multiplication, canonically always considered

as repeated addition, the first factor is called *guṇakāra* (sometimes shortened to *guṇaka* or even *guṇa*), 'the multiplier', the active factor (adjective or adverb depending on whether the second factor functions as a noun or adjective), and the second *guṇya*, 'the multiplied', the passive factor (functioning as a noun or adjective). Since the commutativity of multiplication was understood early, the asymmetry in the nomenclature is mathematically superfluous though not so grammatically: a clear case of the way language impinges on mathematical expression.

32. Our thanks are due to Professor Kamaleswar Bhattacharya and Professor Uma Vaidya for helping us appreciate the grammatical questions at issue in the discussion above.

33. We thank Professor Pierre-Sylvain Filliozat for emphasising this point.

34. The unmissably frequent presence of six fold multiples of powers of 10 not only in this verse but in all of the *Ṛgveda* raises the often suggested possibility of Babylonian influences on vedic decimal enumeration. The material of this section is proof that there is not even a faint vestige of the use of 60 as a base: i) Every single number name makes sense if (and only if) the base is 10. ii) In particular, 60 does not have an independent name but is called 'sixof ten' (and 70 is 'seven of ten' and not, for example, the Sanskrit equivalent of the French '*soixante dix'*). iii) The powers of 60 have no special prominence: 60 itself occurs in its appointed place in the decimally defined sequence of (2.18.5 and 2.18.6) along with the other multiples of 10 and 60^2 = 3,600 is, to our knowledge, not present at all (60^3 is bigger than the biggest number in the *Ṛgveda*). We feel that a more natural explanation for the relative abundance of 6, ..., 60,000 may be sought in the assumed number 360 = 6×60 of days in the year. That particular count of days may have its origin in Old Babylonia. What is certain is that if there was transmission of base-60 place-value counting from Mesopotamia to India, the *Ṛgveda* has no direct evidence of it.

35. The same chapter also lists powers of 10 which, up to *parārdha*, is the same as the one of chapter 5.1 but then continues with other names (ending again with *sarvasmai svāhā*) whose numerical significance is uncertain; Keith ([Ke]) does not read them as number names but some others, for example S. A. S. Sarma, do ([BS]). If they are numbers, the list goes up to 10^{19}. It is tempting to think that *sarva* coming at the end of a long sequence may be a signal that one "cannot name all the numbers". The date of *Taittirīya Saṃhitā* must be well before the earliest *śulbasūtra*. To suggest that the potential infinitude, the *aparimita* nature, of numbers was beginning to be realised soon after *Ṛgveda* is thus not unreasonable.

36. The passage is quoted extensively in [Pl].

37. "Those compounds in which the adjective is a numeral are by the Hindu grammarians treated as a special case called Dvigu" ([Mac1]). "This type, which is called *Karmadhāraya* by the Indian grammarians, is uncommon in the *Saṃhitās*" ([Mac3]).

38. Both in [Ka1] and in a more recent paper [Ka2] which we received after the present work was completed, Kadvany has addressed the question of the relationship between

decimal numbers and grammar. His approach is linguistic/computation-theoretic. The point of our work is that once the recursive construction of based numbers is understood, the only formal machinery needed for the creation of arbitrarily large based numbers is elementary arithmetic together, in the oral context, with a grasp of the principles of word-composition. In that light, taking recourse to formal grammar and modern computation theory to understand the origins of decimal numbers is misleading. It is also ahistorical; 700-800 years before Pāṇini, *Ṛgveda* already has hundreds of non atomic numbers whose names derive from a small set of compounding laws. To present Pāṇini's metalinguistic rules as the direct force behind the genesis of decimal number names is to put the effect before the cause, leading on occasion to conclusions which are empirically untenable ("Perhaps as early as 200 BC [i.e., a millennium after *Ṛgveda*] Indians knew the positional principle..."[Ka2]). Despite this reservation, however, Kadvany's point that the written (symbolic) and the oral (nominal) are equivalent ways of enumeration is one which is in agreement with our general, arithmetically formulated, place-value principle as is the consequent central role of grammatical rules in implementing it. Kadvany does not give many examples to support his thesis and among those he does, there are some which are not factually correct. For instance, *bhūtasaṃkhyā* (his prime candidate for "positional number words") have fundamentally nothing to do with place-values.

39. Murthy ([Mur]) says: "... in the Vedas (in the plural) ... one finds numbers such as 1, 3, 7 and 10 ... at least on a few hundred or perhaps even more than a thousand occasions ...". He has a selected short list of numbers to which he attributes a symbolic significance. Sarma (in [BS]) mentions a few number names from *Ṛgveda* (mostly atomic numbers and powers of 10) and does say ". . . the decimal numeration adumbrated in the Vedic literature is as perfect in its fundamentals as the numeral system of modern times" but there are also statements which run counter to the actual evidence from *Ṛgveda*. Neither of these works is concerned with how a given number name determines a unique number.

Bibliography

[Ba] Bhagyashree Bavare, "The Origin of Decimal Counting: Analysis of Number Names in the *Ṛgveda*", *Gaṇita Bhārati* 32 (2010) 25-35

[Bi] Madeleine Biardeau, *Bhartṛhari*. *Vākyapadīya Brahmakāṇḍa avec la vṛtti* de *Harivṛṣabha* E. de Boccard, Paris, 1964

[BS] A. K. Bag and S. R. Sarma (Eds.), *The Concept of Śunya*, Indira Gandhi International Centre for the Arts, Indian National Science Academy and Aryan Books International, New Delhi, 2003

[Bu] E. Burgess, *The Sūryasiddhānta*, (reprint), Motilal Banarsidass, New Delhi, 1989

[Co] H. T. Colebrooke, *Algebra, with Arithmetic and Mensuration, from the Sanscrit of Brahmegupta and Bhascara*, Murray, London, 1817

[Di1] P. P. Divakaran, “Notes on *Yuktibhāṣā*: Recursive Methods in Indian Mathematics” *Studies in the History of Indian Mathematics*, C. S. Seshadri (ed.), Hindustan Book Agency New Delhi, 2010

[Di2] P. P. Divakaran, “Birth of Calculus with Special Reference to *Yuktibhāṣā*” (originally titled “Notes on *Yuktibhāṣā*: the Birth of Calculus”), *IJHS*, 47.4 (2012) 771-832

[DO’F] Wendy Doniger O’Flaherty, *The Rig Veda. An Anthology* (Indian edition), Penguin Books, New Delhi, 2000

[DS] Bibhutibhushan Datta and Avadhesh Narayan Singh, *History of Hindu Mathematics* (reprint) Bharatiya Kala Prakashan, Delhi, 2001

[Fi] Pierre-Sylvain Filliozat, “Modes of Creation of a Technical Vocabulary: the Case of Sanskrit Mathematics”, *Gaṇita Bhārati* 32 (2010) 37-53

[Gi] Brendon S. Gillon, “*Pāṇini’s Aṣṭādhyāyī* and Linguistic Theory”, *J. Ind. Philos*. 35 (2007) 445-468

[Go1] J. Gonda, *Triads in the Veda*, North-Holland, Amsterdam, 1976

[Go2] J. Gonda, *Vedic Ritual. The Non-Solemn Rites*, E. J. Brill, Leiden, 1980

[Gr] Ralph T. H. Griffith, *The Hymns of the Ṛgveda* (reprint) Motilal Banarsidass, Delhi,1999

[Gu] R. C. Gupta, “Spread and Triumph of Indian Numerals”, *IJHS*, 18.1 (1983) 23-38

[Ha] Takao Hayashi, *The Bakhshali Manuscript*, Egbert Forsten, Groningen, 1995

[Iy] K. A. Subramania Iyer, *The Vākyapadīya of Bhartṛhari with the vṛtti*, Chapter I, Deccan College, Poona, 1965

[Ka1] John Kadvany, “Positional Value and Linguistic Recursion”, *J. Ind. Philos.* 35 (2007) 487-520

[Ka2] John Kadvany, “Indistinguishable from Magic: Computation is Cognitive Technology”, *Minds and Machines* 20 (2010) 119-143

[Ke] A. B. Keith, *The Veda of the Black Yajus School entitled Taittiriya Sanhita* (2 vols.), Harvard University Press, Cambridge, 1914

[Mac1] Arthur A. Macdonell, *A Sanskrit Grammar for Students* (Indian edition) Motilal Banarsidass, Delhi, 1974

[Mac2] A. A. Macdonell, *Vedic Grammar for Students* (reprint) Motilal Banarsidass, Delhi, 1993

[Mac3] A. A. Macdonell, *Vedic Grammar* (reprint) Munshiram Manoharlal, New Delhi, 2007

[Mat] B. K. Matilal, *Perception: An Essay on Classical Indian Theories of Knowledge*, Oxford University Press, New Delhi 1986

[Mum] David Mumford, "What's so Baffling About Negative Numbers? – a Cross-Cultural Comparison" *Sudies in the History of Indian Mathematics*, C. S. Seshadri (ed.) Hindustan Book Agency, New Delhi 2010

[Mur] S. S. N. Murthy, "Number Symbolism in the Vedas", *Electronic J. Vedic Studies* 12 (2005) 87-99

[M-W] M. Monier-Williams, *A Sanskrit English Dictionary* (reprint) Motilal Banarsidass, Delhi, 2005

[Na] Roddam Narasimha, "The *Yukti* of Classical Indian Astronomical Science" *Foundations of Science*, B. V, Sreekantan (ed.), PHISPC, New Delhi, 2007

[Pi] K. Raghavan Pillai, *The Vākyapadīya. Critical Text of Cantos I and II*, Motilal Banarsidass, Delhi, 1971

[Pl] Kim Plofker, *Mathematics in India*, Princeton University Press, Princeton, 2009

[Re] "Hymnes á Agni" in Louis Renou, *Etudes védiques et Paṇinéennes* (vol. 13) E. de Boccard, Paris, 1964

[Sa] K. V. Sarma, *Gaṇita-Yukti-Bhāṣā* of Jyeṣṭhadeva (vol. 1), Hindustan Book Agency, New Delhi, 2008

[SB] S. N. Sen and A. K. Bag, *The Śulbasūtras of Baudhāyana, Āpastamba, Kātyāyana and Mānava,* Indian National Science Academy, New Delhi, 1983

[St1] J. F. Staal, *Word Order in Sanskrit and Universal Grammar*, Kluwer, Dordrecht, 1967

[St2] Frits Staal, "The Sanskrit of Science", *J. Ind. Philos*. 23 (1995) 73-127

[St3] Frits Staal*, Ritual and Mantras: Rules Without Meaning*, Motilal Banarsidass, Delhi, 1996

[St4] Frits Staal, "Vedic Mudras" in *Agni: The Vedic Ritual of the Fire Altar* (vol. II), Frits Staal (Ed.) (Indian edition), Motilal Banarsidass, Delhi, 2001

[St5] Frits Staal, "Artificial Languages between Innate Faculties", *J. Ind. Philos*. 35 (2007) 577-596

[St6] Frits Staal, *Discovering the Vedas*, Penguin Books, New Delhi, 2008

[St7] Frits Staal, "On the Origins of Zero", *Studies in the History of Indian Mathematics*, C.S. Seshadri (ed.), Hindustan Book Agency, New Delhi, 2010

[TA] Rama Varma (Maru) Tampuran and A. R. Akhilesvara Ayyar, *Yuktibhāṣā* (Part 1), *General Mathematics* (in Malayalam) Manglaodayam Ltd. Trichur, 1948

[Th] G. Thibaut, "On the Śulvasūtras", *J. Royal Asiatic Soc. Bengal* 44.1

[Va] Śrīśa Candra Vasu, *The Siddhānta Kaumudī of Bhaṭṭoji Dīkṣita,* vol. I, (reprint) Motilal Banarsidass, Delhi, 1977