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BHĀSKARA I'S VERSIFIED SOLUTIONS OF A LINEAR INDETERMINATE EQUATION

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Abstract

At the end of Bhāskara's 7th century commentary on the mathematical part of the *Āryabhaṭīya* 91 verses provide solutions of linear indeterminate equations using astral parameters. The number of civil days and the number of revolutions elapsed since the beginning of the great *yuga* are computed knowing specific mean positions of the Sun, the Moon, the Moon's Apogee, the Moon's ascending node, Mars, Mercury, Jupiter, Venus, Saturn, and the Moon's Anomaly. Other computed elements concern intercalary months, intercalary days, omitted days and the Rsine of the sun's declination. This article provides an English translation of the verses, an explanation and exploration of the data provided. I propose these tables had two aims: first to be used as a tool when solving more elaborate equations; and second as an exploration of the ratios between planet positions and time elapsed, if not a more general exploration of the numbers involved in these ratios. This article also shows how the verses are used to tabulate data in a compact form.

Key words: Āryabhaṭa, Bhāskara, *Kuṭṭākāra*, Table of solutions of indeterminate equations, Syntactical numerical table.

To Aurelie, who thank god is not a number in a table!

1. INTRODUCTION

At the end of Bhāskara's 7th century commentary on the mathematical part of the *Āryabhaṭīya* (499 AD), 91 verses provide the solutions of linear indeterminate equations using astronomical constants. These verses exist in all the manuscripts used by K. S. Shukla's 1976 edition of Bhāskara's *Āryabhaṭīyabhāṣya* (Shukla 1976, pp. 156-171). They are translated at the end of this article, in Appendix A. They constitute, to our knowledge[1], the only collection of equation solutions in Sanskrit mathematical and astral literature.

Why and how then were these solutions collected? Who could have used them, and why did they not have any posterity? These are the questions that we will try to elucidate in the following.

The *Āryabhaṭīya* is made of four[2] parts (*pāda*): the first coins parameters and is named after the meter in which it is composed, the *gītikā*. The second part is on mathematics (*gaṇita*), the third on time computation (*kāla-kriyā*) and the fourth on the celestial sphere (*gola*). Bhāskara's commentary follows the order of Āryabhaṭa's text, commenting his verses more or less one after the other[3].

The verses studied here are provided within Bhāskara's 7th century commentary. They are situated at the end of the commentary on the mathematical part and before the commentary on time computations. They are in this way situated

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at the interface between mathematics (*gaṇita*) and astral science (*jyotiṣa*).

2. Betwixt mathematics and astronomy

Indeed, Bhāskara at the opening of his commentary stresses the complexity of relationships between the *Āryabhaṭīya*'s different parts[4]: planetary computations can be considered a sub-part of mathematics; but here mathematics is a small part of a treatise devoted to planetary and time computations[5]. Thus examining these collected solutions is also a way of reflecting on the relationship of mathematics and astral science for Bhāskara.

Actually, these verses concern a kind of problem and a method of resolution, which later in Sanskrit literature, will be famed for its ambiguous position in classifications of mathematics[6] and astral science: the pulverizer (*kuṭṭāka*)[7]. In Bhāskara's commentary, the pulverizer is classified with proportion (*anupāta*) among specific topics of mathematics (*gaṇita-viśeṣa*)[8]. Planetary computations (*graha-gaṇita*) are similarly characterized as belonging to such specific topics. In this part of the commentary Bhāskara does not elucidate how both could be related. However, in practice, with trigonometry, the procedural form of the Pythagorean Theorem and the Rule of Three, pulverizers appear as a type of problem and procedure useful both in mathematical astronomy and in mathematics. The pulverizer is treated by Āryabhaṭa at the end of the mathematical part. In Bhāskara's commentary of these last verses of the mathematical part, examples both purely numerical and using astronomical parameters are solved[9].

This article then will explore the mathematical and astral interpretation of the numbers coined in these verses, in an effort to understand what was their aim.

However, another question is raised by these verses: Is this but another instance of the kind of numerical table used in India before the tabular format with lines and columns took over? Do these verses testify to a way of stocking data in a condensed form that provides tabular material? Additionally, can the verses be seen also as a sort of formal table, a syntactical equivalent of rows and columns?

Before turning to these questions we will first explore how these solutions were found.

A first section of this article evokes briefly the mathematical procedure and the astral backdrop enabling the computation of these values. We will then speculate on what could have been the aim of these verses. A last section reflects on the tabular nature of the data and verses considered.

3. Understanding the mathematics, Understanding the astronomy

This section aims at explaining first how the numbers provided in these verses were obtained, how they could have been used mathematically, and finally what they could have meant from an astral point of view. The numerical data recorded in all 91 verses, translated in Appendix A, is tabulated in Appendix B at the end of this article. In verses 32 and 33, the last of the mathematical part of the *Āryabhaṭīya*, a rule is given to solve a set of problems grouped by Bhāskara under the name *kuṭṭākara*[10]. The pulverizer without remainder (*nir-agra-kuṭṭāka*) solves the indeterminate equation, for $a,b,c,x,y \in \mathbf{N}^*$ $y=(ax\pm c)/b$ (1). x and y are unknown, a, b and c are given fixed constants. The problem as presented in (1), its solution, and the method used to solve it will be noted KU(a, b, c)[x,y][11].

For (i) the problem, (ii) the general method used to solve the problem, and (iii) the first solution, α, of the couple (α, β), Bhāskara uses the expression *kuṭṭākara*. In the verses studied here x is more often called *guṇakāra* "multiplier". Several synonyms of multiplier are subsequently used, such as *guṇa*, or "multiplicand", *gunya*, and so forth. *y* is usually called "the result'' (*labdha, lābha, labdhaka etc.*), translated here as "quotient'' because it is the result of a division.

Studies of the different steps of Āryabhaṭa's pulverizer, as understood by Bhāskara, have been published several times already[12]. The verses studied here contain ordered pairs of solutions of what in secondary sources is generally called a "permanent pulverizer" (*sthira-kuṭṭāka*). A "permanent pulverizer'', rests on the fact that if (α, β) is a solution of KU(a, b, - 1)[x, y], then (α_0,β_0) is a solution of KU(a, b, - c)[x, y], for $c \in \mathbf{N}^*$, were α_0 and β_0 are respectively the remainders of the division of $c\alpha$ by b and $c\beta$ by a[13]. Indeed, KU(a, b, - 1)[x, y] has solutions according to Bezout's identity only if a and b are coprime. Consequently, to solve such a problem, the first movement will be to reduce the given values (a_0, b_0) of the problem to ensure its solvability. Further also because of the Bezout theorem, for $p \in \mathbf{Z}$, if (α, β) solution of KU(a, b, - 1)[x,y], then (α_0, β_0), where $\alpha_0 = c\alpha + bp;\ \beta_0 = c\beta + ap$ is a couple solution of KU(a, b, -c)[x,y][14]. Thus with solutions of KU(a, b, - 1)[x, y], solutions of KU(a, b, - c)[x, y] are easily generated. So that computing solutions of KU(a, b, - 1)[x, y] can be seen as a short-cut to computing solutions of KU(a, b, - 1)[x, y]. This is one of the key ideas behind the use of the data coined in the 91 verses.

A graphic representation of the function f(x)= (ax±c)/b is a straight line. Integer solutions of KU(a, b, - c)[x, y] can then be represented as an infinite set of dots on a given line. For each changing value of c, corresponds a new straight line, parallel to others for a same pair of (a,b). In this context one can then fix the smallest positive integral solution, which is computed by means of the pulverizer process described in the Ab and Bhāskara's commentary. This is illustrated in Fig. 1.

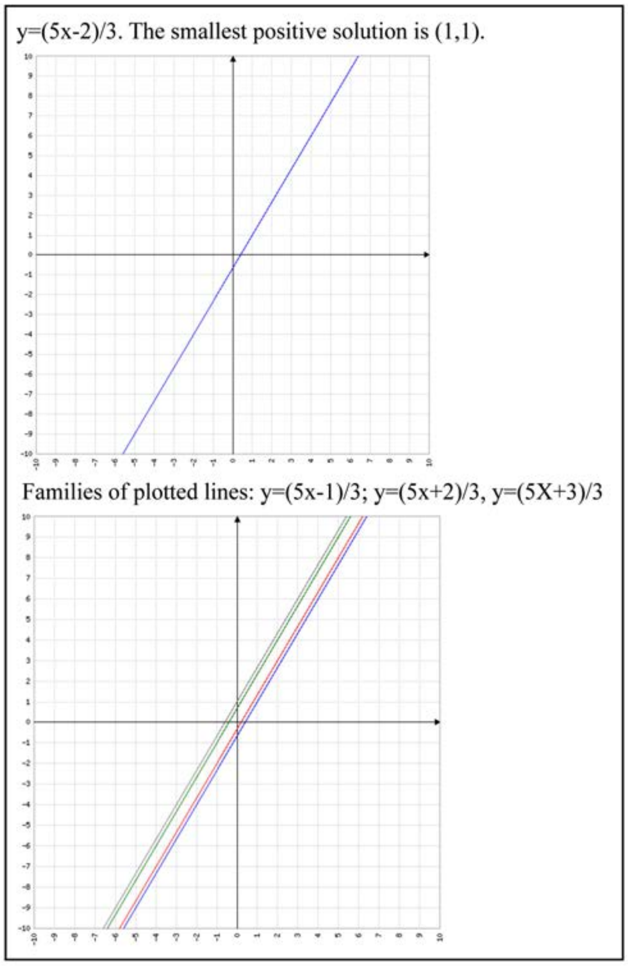

Fig. 1: Graphical representation of KU(5,3,c)[x,y]

Bhāskara does not have a specific name for the "permanent pulverizer'', which is included as a sub procedure in the larger category of "pulverizer without remainder''. He introduces this short-cut in his commentary on example 7 of BAB.2.32-33 and in many subsequent examples[15]. Thus, we can deduce that Bhāskara considered this "short-cut'' as a convenient sub-procedure while solving pulverizer type problems, with non trivial values of *c*.The numerical couples transmitted in the 91 verses studied here are solutions (α,β) of KU$(a, b, -1)[x, y]$, for implicit astronomical parameters *a* and *b*.

Let us take the data given in BAB.Ku.9. The verse runs as follows[16]:

"शीतरश्मि[भगणैः प]रकुट्टिते सप्तरामवसुषट्स्वराद्रयः।
लब्धराशिनिचयः व्यवस्थितः पुष्कराग्निकृतनागबाहवः॥९॥

> When pulverized [by the revolutions] of the moon, <the mulitplier is made of the numbers> seven-three-eight-six-seven-seven [776 837]|
>
> A heap which is a quotient quantity is laid out as <the following numbers> three-three-four-eight-two [28 433] ||9||"

We will set aside here how the numbers are expressed in the verses- this will be treated in the next section. This verse provides two values, those forming the couple solution (α,β): (776 837, 28 433). The values of *a* and *b* are not stated explicitly. But for reasons that will be explained later, we know that this couple is a solution of the following equation: y= (78 898*x*-1)/2 155 625.

Later literature notes that if KU(a,b,c)[α_0, β_0], then KU(a,b,c)[α_0+bk, β_0 +ak], for $k\in \mathbf{Z}$ [17]. This property enables one to derive infinite solutions of such indeterminate problems. This property is not stated by Āryabhaṭa or Bhāskara. However it seems to be used in one case here. Indeed, verse 14 provides two values, (6 518, 18 550 831 927), solutions of y= (9 816 173 568*x*-1)/3 449

> "तत्परेषुधृतिभूतषट्ककाः निर्दिश्न्ति गुणकारसंख्यया।
> ऋक्षनन्दशशिरामकुञ्जरव्योमबाणशरधार्तवोऽपरः॥१४॥
>
> BAB.Ku.14 In thirds, with a number which is the multiplier eighteen-five-six [6 518] indicate <the multiplier>|
>
> The following (i.e. the quotient) is <made of the digits> twenty seven-nine-one-three-eight-zero-five-five-eighteen [18 550 831 927]||14||"

However the smallest positive solution is: (3 069, 8 734 658 359)=(α_0, β_0)[18].

Since α_0 + b=3 069+ 3 449= 6518

and β_0 +a = 8 733 658 359 + 9 816 173 568 = 18 550 831 927.

The information contained in verse 14, can be formalized as:

KU(9 816 173 568, 3 449, -1)[6 518, 18 550 831 927]=K(a,b,-1)[α_0+b, β_0 +a].

This is the only case in which the couple solution is not the smallest possible.

How then do we know the values of *a*, *b*, and *c* that are implicit in theses verses? The answer is to be found in the astral backdrop within which these couples of numbers are given.

4. Planetary Pulverizers

Problems solved by a pulverizer can be derived from constant ratios between circular planetary movements and passing time. Theoretically an astral interpretation of the couple solutions of a pulverizer KU(a_g, *b*, *-1*)[*x*, y_g] is the following: knowing that a_g revolutions are performed by planet *g* in *b* days, when *g* is in λ_g, then *x* days have elapsed, and y_g revolutions have been performed by *g*. In other words, x is the number of days that g needs to perform y_g revolutions.

Indeed, in the verses considered here, as when it is introduced in the *Āryabhaṭīyabhṣya*, the *Mahābhāskarīya*, or the *Laghubhāskarīya*, the *kuṭṭākara* is used with astronomical parameters. Bhāskara provides a name for such a category of pulverizers: a "planet-pulverizer'' (*graha-kuṭṭākara*)[19]. The 91 verses can be seen as belonging to this category of pulverizers. The relation between verses and astral parameters are represented in Table 1.

The astronomical basis of these planetary pulverizers is quite standard and has been often described[20]. For Āryabhaṭa there exists an overall time cycle, a *(māha)yuga* ("great yuga") which lasts 4 320 000 years (taken standardly as having began on February 18, 3102 BC at sunrise at the intersection of the equator and the meridian in Ujjain). Each planet makes a fixed integer number of revolutions in a *yuga*. This number is usually called in the secondary literature, the planet's "revolution number''. If a_g is the number of revolutions of given planet *g* (*graha*), during the time cycle of the *yuga*, and *b* the number of civil

Table 1: Data in Verses (1)

Verses	Theme
1-2	Introduction
3-8	Revolutions of the Sun since the beginning of a *yuga*
9-16	Revolutions of the Moon
17-24	Revolutions of the Moon's Apogee
25-32	Revolutions of the Moon's Ascending node
33-40	Revolutions of Mars
41-48	Revolutions of (the *śīghrocca* of) Mercury
49-56	Revolutions of Jupiter
57-64	Revolutions of (the *śīghrocca* of) Venus
65-72	Revolutions of Saturn
73-80	Revolutions of the Moon's Anomaly
81	Intercalary Days since the beginning of a *yuga*
82	Omitted Days
83	Sun's Declination
84-91	Intercalary Months since the beginning of a *yuga*

days in the *yuga* (these are fixed parameters), and if y_g is the total mean motion of g during a certain time x- evaluated in terms of civil days. The assumption is that $a_g /b= y_g /x$. A planet does not necessarily make an integer number of revolutions but has, in addition to an integral value, a fractional part, or residue (*śeṣa*). Usually only this fractional part of y_g is known, it is what can be observed through an angular measure in the sky: λ_g, the mean longitude of the planet. $y_g = y + \lambda_g$ The remaining part, y, is unknown, together with x.

Thus $a_g/b = (y+ \lambda_g)/x$ and therefore $y= (a_g x-b \lambda_g)/b$ in which we recognize equation (1).

Note that if $c = - b \lambda_g$, then $\lambda_g = - c/b$

Ab.1.3-4 gives the numbers of revolutions of the sun, moon, earth etc. in a *yuga*, and the date of its beginning[21]. Civil days are defined in Ab.3.5c[22].

Table 2 lists the parameters which are subsequently used by Bhāskara in the solved examples of his commentary of Ab.2.32-33 and in the 91 verses.

Table 2: Astral parameters used by Bhāskara I

Parameters for a whole Yuga	Notation	Values as in Ab.1.3-4; Ab.1.6
Civil days	b	1 577 917 500
Sun's revolutions	a_s	4 320 000
Moon' revolutions	a_c	57 753 336
Moon's Apogee's revolutions	a_{cu}	488 219
Moon's Ascending Node's revolutions	a_p	232 226
Mars' revolutions	a_b	2 296 824
(*Śīghrocca* of) Mercury's revolutions	a_{bu}	17 937 020
Jupiter's revolutions	a_{gu}	364 224
(*Śīghrocca* of) Venus's revolutions	a_{bh}	7 022 388
Saturn's revolutions	a_{sa}	146 564
Moon's Anomaly's revolutions	a_{ck}	57 265 117
Intercalary days	a_{ad}	47 800 080
Solar months	b_{ad}	51 840 000
Omitted days	a_{av}	25 082 580
Lunar days	b_{av}	1 603 000 080
RSine of the maximum of the sun's declination	a_{apa}	1397
Radius of the sky	b_{apa}	3438
Intercalary months	a_{am}	1 593 336

Take for instance the parameters used in verses 9 to 16 concerning the revolutions of the Moon during a *yuga*. With Ab.1.3, we know that the revolutions of the moon (*candra*), a_c, during a whole *yuga* has the value 57 753 336. And the number of civil days during a *yuga*, b, is equal to 1 577 917 500.

These are the two underlying parameters of the pulverizers solved in the series of verses, 9 to 16.

Values of a_g can vary according to the measuring units considered for a revolution. These measuring units, as seen in Table 3, naturally recover the different subdivisions of a circle in mathematics. Therefore in verses 9 to 15 the verses provide solutions for pulverizers considering values of a_c in revolutions (verse 9), signs (verse

Table 3: Subdivisions of a Revolution

Sanskrit	English	Amounts in a Revolution	Powers of 60
maṇḍala	Revolutions	1	1/60 x 60
rāśi	Signs	12	1/5 x 60
bhāga	Degrees	360	6 x 60^2
liptā	Minutes	21 600	6 x 60^3
vikalā	Seconds	1 296 000	6 x 60^4
tatparā	Thirds	77 760 000	6 x 60^5
pratatparā	Fourths	4 665 600 000	6 x 60^6

10), degrees (verse 11), minutes (verse 12), seconds (verse 13), thirds (verse 14) and fourths (verses 15). Note that the use of such fine subdivisions as thirds (*tatparā*) and fourths *(pratatparā*) is an indication of how accurate the values considered in these verses are[23].

However, values of a_g are further transformed because before entering the pulverizer process, common divisors of a_g and b are used to reduce these two numbers. Thus, the values considered for a_c are actually reduced by divisors that are listed in the last verse of the series, in verse 16:

अंशातारभ्यशीतांशोः पञ्चपञ्चगुणः परः।
छेदः कल्प्यः क्रमातत्रदन्तशैलसमन्वितः॥.:

BAB.Ku.16 Starting with the moon's degrees the divisor should be set here in due order as five [5] five [5] after five [5] with thirty-two-seven [732] ||16||

As shown in Table 6, in verse 9, both a_c and b are divided by 732.

Since a_c /732= 57 753 336/732= 78 898

and b/732= 1 577 917 500/732= 2 155 625

then the numerical data coined in verse 9 can thus be stated as

KU(78 898, 2 155 625, -1) [776 837, 28 433]. In other words, (776 837, 24 833) is one couple solution of y=(78 898*x*-1)/2 155 625.

These results have a meaning, from the point of view of the astral parameters they use, which is the following: Knowing that during 2 155 625 days, the moon performs 78 898 revolutions, when it is known that the fractional part of a revolution 1/2 155 625 has been performed by the Moon, then for $k \in$ N, 776 837 + 2 155 625k days have elapsed since the beginning of a *yuga* and 28 433 + 2 155 625k + 1/2 155 625 revolutions have already been completed by the Moon.

In other words, in these verses KU(a_c,b,-1)[x, y_c] are considered. y_c+1/b corresponds to the mean motion in revolutions (or signs, degrees, minutes, seconds, thirds and fourths) of the moon since the beginning of the current great *yuga* and x the number of civil days elapsed since the beginning of the *yuga*. This information is the one that enables us to unravel the data on which the verses considered rest. Further it can give us a clue to the way these values were interpreted and hypothetically used.

5. Speculating on the uses and meaning of the values obtained

Solving other equations

Most of the solution-couples presented here, can be seen as intermediary resolutions for a problem of the form KU(a, b, ±c)[x, y]. Indeed, this is how such a pulverizer method is introduced by Bhskara in his commentary on these rules. In this sense, and because the verses appear in the commentary of the mathematical part of the *Āryabhaṭīya*, this set of verses can be seen as a mathematical tool to be used when computing solutions of other equations.

However, most of these solutions, can have an astronomical interpretation, linking the position of a planet to the time elapsed.

6. Linking Time and Space

As we have seen, the ratios linking periodical planetary movements in the sky and passing time- a topic traditionally explored by ancient mathematical astronomy- enabled the formulation of problems that could be solved by a pulverizer.

The astral interpretations of the pulverizer concern both time and position. Some of the entities considered are not planets per se, but theoretical points associated to the movement of planets. They can however be considered as planet entities. This is the case for Mercury and Venus's *śīghrocca*, the Moon's apogee and the Moon's ascending node, all noted in Ab.1.3 and Ab.1.4. The Moon's anomalistic revolutions, similarly can be treated in this way. The last cases, ratios of Rsine of longitude and declination set aside, seem to articulate how lunar and civil days and months, omitted or intercalated, are also in proportion.

In all of these cases then, whether what is computed can be interpreted as real planetary movements or theoretical points in the sky, related to planetary movements or to time elapsed, an interpretation of the results of the pulverizer is possible, although the accuracy of the values questions how "practical'' such values could be. Was the aim of the pulverizer than a way of accessing to very accurate results? Another, hypothesis, could be of course that such astral parameters were used for exploring the numbers at hand, outside of their astral meaning.

The various explorations by the versificator(s) of the relations between computation, measure, value and numbers, may also be an expression of this question itself: should such numbers be taken for something else than numerical values?

7. Numbers in Verses and Numerical Tables

The 91 verses that end Bhāskara's commentary on the mathematical part of the *Āryabhaṭīya* are referred to by Shukla as "tables''[24]. Is any string of numbers a kind of numerical table? In the following we will suggest some characterizations of numerical data in relation to tables, before looking at the kind of textual object with numbers created by these verses.

7.1 Numerical Tabular data

Inspired by an algorithmic and functional definition of what is a numerical table, in the following we will call "tabular data'' any set of two varying increments producing one or several outputs. This may be distinct from a "tabular alinement'' or a "numerical array'' involving lines and columns[25].

As seen in Fig. 2, KU(a,b,c)[x,y] has three increments a, b and c, producing two outputs, *x* and *y*.

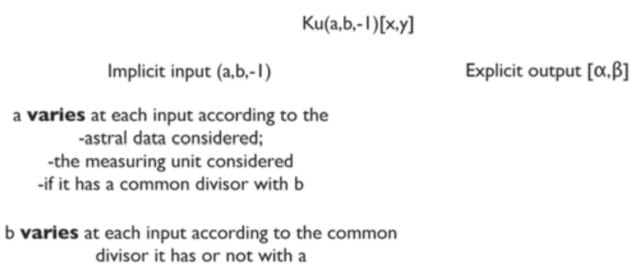

Fig. 2. Inputs and Outputs

As noted previously, values of *a* and *b* are not explicitly stated in the verses considered here. In this sense, as in other versified tables known in Sanskrit literature, the increment of the tables is implicit. In practice, values of *b* and *c* are previously fixed in the verses here.

Indeed, *c* has a constant value: -1.

b is the constant number of civil days in a *yuga*, 1 577 917 500.

Nonetheless, as we have seen, the value of *b* is constantly changing because before solving the equation, *a* and *b* are reduced by common divisors.

a varies in three ways:

a, in most of the cases- this nuance will be discussed below-, is the number of revolutions of a planet. It varies then every set of eight verses, as seen in Table 1, with values presented in Table 2. Thus the 91 verses can be seen as containing 11 or 12 subsets of numerical tabular data. Indeed, for each astral parameter incremented in *a*, two other variations take place.

a also varies because the measuring unit used to evaluate the number of revolutions of a given planet changes from revolutions, to signs, degrees, minutes, seconds, thirds and fourths, in this regular order, as shown in Table 3.

Finally, each sub-set of *a* also varies because it can be reduced with *b* by a common divisor, *d*. *d* can vary with every new value incremented into *a*.

These many variations of *a* and *b* which in turn produce different couples (α, β) are fully exposed in numerical arrays in Appendix B. They explain why we can consider that different sets of tabular data are coined in these verses: two varying increments produce two outputs.

However I would like to argue that something else is at play here as well.

Indeed, in the following, in an attempt to explore whether this numerical tabular material is transmitted through "syntactical tables'' two questions will be raised. First, is the data given in a strict order, which could be the syntactical equivalent of the order of rows and columns? Second, is the data retrieved without requiring necessarily a linear "reading''?

BAB.Ku.2 states

कुट्टाकारः च लाभः च द्वन्द्वतः भगणादितः|
निर्दिश्यते क्रमात त्रतद्विदाम्प्रीतयेमया||2||॥॥

> I indicate here with delight for those who know them the pulverizer (*kuṭṭākara*) and quotient (*lābha*) <derived> two by two from revolutions and so forth in due order ||2||

Verse 2 provides once and for all information both on the order (*kramāt*) in which the data (*kuṭṭākara* in the sense of multiplier and quotient) is given in the verse and more generally on how the data is arranged in respect to the implicit ordered variations of the parameters.

7.2 Order

The verses indeed list ordered pairs of numbers, they thus list couples of solutions (with our notations (α, β) of KU(a,b,-1)[x, y).

Take for instance, verse 9 again:

शीतरश्मि[भगणैः] प्रकुट्टिते सप्तरामवसुषट्स्वराद्रयः।
लब्धराशिनिचयः व्यवस्थितः पुष्कराग्निकृतनागबाहवः॥ः

> When pulverized [by the revolutions] of the moon, <the mulitplier is made of the numbers> seven-three-eight-six-seven-seven [776 837]|
> A heap which is a quotient quantity is laid out as <the following numbers> three-three-four-eight-two [28 433].||

As indicated by the brackets, the fact that the first value given here is *a* "multiplier'' (α) is assumed to be known, since this was already specified in Verse 2. This *rathoddhatā* verse provides the two numbers α and β in this order. All verses always state these values in this order, whatever the meter.

The text itself refers regularly to this order. Indeed, the data is enumerated "in due order" (*kramāt*, *krameṇa*). Thus, in verse 14 the element coming after the multiplier is called *apara*, "the subsequent''. Its place in the order is thus referred to rather than its nature as a quotient.

Further, verse 2 alludes to the variations of *a*: first as one of the two components enabling the derivation of α and β (*dvandva*). This order is then integrated into the second larger order of the sub-sets of numerical data provided in the verses. Thus, as seen in Table 1 in most cases, groups of seven verses, give solution couples for a certain astral component of *a*. Verse 9 to 16 list elements concerning revolutions of the moon, verses 84 to 91 on intercalary months. The seventh verse, lists the divisors used to reduce *a* and *b* for specific equations, as verse 16 lists the reducers of a_c, verse 91 those of a_{am}[26].

Verse 2 also alludes to another variations of *a* by suggesting that its measuring will vary in a given standard order (from revolutions and so forth, *bhagaṇādi*).

In verse 15, the measuring unit (fourths coming after thirds), and the quotient (coming after the multiplier) are referred to by the fact that they come after what precedes, if it is not that the units of the number enumerated need to be stated in the right order (*kramāt*) as well

> तत्परेषु परतः च कीर्तिताः रामनन्दयमदस्रकाः क्रमात् ।
> रुद्रबानगिरिदस्रसागराः भूतषट्कतिथिनन्दविक्रमाः ॥

> In what follows thirds (e.g fourths, *tatpareṣu parataḥ*) three-nine-two-two are mentioned [2 293] in due order|
>
> In due order, eleven-five-seven-two-four five-six-fifteen-nine-three [3 91 565 427 511] ||15||

That the order of the measuring units for *a*, is considered known is particularly clear in the *anuṣṭubh* which provide the reducers according to measuring unit.Thus, for example, in verse 91, after signs, all the other measuring units are referred to as 'the remaining' (*śeṣa*).

However, there is a certain amount of inaccuracy in these *anuṣṭubh*s, since the loose enumeration does not always apply to all its terms. For instance, in verse 48 dealing with the divisors used for computing solutions from astral data concerned with (the *śighrocca* of) Mercury, five is announced as a divisor for "degrees etc.''[27]. As seen in Table 10 only degrees, minutes and seconds are reduced by five; thirds and fourths are left untouched.

Therefore the order in which the data is provided is structured by its implicit data. The structure of the successive verses rests on regular variations of its implied increments: order and conciseness are thus closely intertwined. Further, the order of the data is also structured by the verses and its meter. This great regularity which uses the versification is encouraging if we want to consider these verses as not only containing "tabular data'' but also as a kind of "syntactical table''. Indeed, with a regular order, emplacement and versification, one can imagine that the values could be retrieved without a statement (learnt by heart) or reading of all the verses.

7.3 Arranging Data within verses

The verses (*ślokas)* considered here are quite diverse[28]. They are in their great majority the *triṣṭubh* (*indravajrā* or *upajāti*). The *anuṣṭubh*, with 8 syllables per quarter, is used to indicate values of *d*, the common divisor of *a* and *b*, every set of seven verses. Thus, as seen in Table 4, the meter change after each seven groups, provides a syntactical form indicating the structure of the data, isolating each group of seven verses.

The first twenty verses display a great variety actually. Verse 17 is a sort of hybrid, starting as an 11 syllable *upajāti* and ending as a 14 syllable *vasantatilakā*. It marks a clear turning point within the metre structure of the verses, because until the case points of verses 81-83, this will be a regular alternance of seven *triṣṭubh* (*indravajrā* or *upajāti*) and an eight *anuṣṭubh*. However, the accuracy with which this order is used can be wanting at times: as seen in Table 12 not all verses follow the pattern of one value per half-meter. Further, as seen in Table 1 the last 10

Table 4: Data in Verses

Verses	Theme	Meter
1-2	Introduction	*anuṣṭubh*
3	Revolutions of the Sun in revolutions and signs	*praharṣiṇī*
4-5	Revolutions of the Sun in degrees, minutes, seconds	*vasantatilakā*
6	Revolutions of the Sun in thirds	*rathoddhata*
7	Revolutions of the Sun in fourths	*vasantatilakā*
8	Common divisors for a_s and b	*anuṣṭubh*
9-15	Revolutions of the Moon	*rathoddhatā*
16	Common divisors of a_c and b	*anuṣṭubh*
17	Revolutions of the Moon's Apogee	half *upajāti*, half *vasantatilakā*
18-20	Revolutions of the Moon's Apogee in signs, degrees and minutes	*indravajrā*
21-22	Revolutions of the Moon's Apogee in seconds and thirds	*upajāti*
23	Revolutions of the Moon's Apogee in fourths	*indravajrā*
24	Common divisors of a_{cu} and b	*anuṣṭubh*
25-26	Revolutions of the Moon's Ascending node in signs and degrees	*indravajrā*
27	Revolutions of the Moon's Ascending node in minutes	*upajāti*
28-31	Revolutions of the Moon's Ascending node in seconds, thirds and fourths	*indravajrā*
32	Common divisors of a_p and b	*anuṣṭubh*
33	Revolutions of Mars	*upajāti*
34-36	Revolutions of Mars in signs, degrees and minutes	*indravajrā*
37	Revolutions of Mars in seconds	*upajāti*
38	Revolutions of Mars in thirds	*indravajrā*
39	Revolutions of Mars in fourths	*upajāti*
40	Common divisors of a_b and b	*anuṣṭubh*
41	Revolutions of (the *śīghrocca* of) Mercury	*upajāti*
42-46	Revolutions of (the *śīghrocca* of) Mercury in signs to thirds	*indravajrā*
47	Revolutions of (the *śīghrocca* of) Mercury in fourths	*upajāti*
48	Common divisors of a_u and b	*anuṣṭubh*
49	Revolutions of Jupiter	*indravajrā*
50	Revolutions of Jupiter in signs	*upajāti*
51-52	Revolutions of Jupiter in degrees and minutes	*indravajrā*
53	Revolutions of Jupiter in seconds	*upajāti*
54	Revolutions of Jupiter in thirds	*indravajrā*
55	Revolutions of Jupiter in fourths	*upajāti*
56	Common divisors of a_{gu} and b	*anuṣṭubh*
57-58	Revolutions of (the *śīghrocca* of) Venus in revolutions and signs	*indravajrā*
59-60	Revolutions of (the *śīghrocca* of) Venus in degrees and minutes	*upajāti*
61-62	Revolutions of (the *śīghrocca* of) Venus in seconds and thirds	*indravajrā*
63	Revolutions of (the *śīghrocca* of) Venus in fourths	*upajāti*
64	Common divisors of a_{bh} and b	*anuṣṭubh*
65	Revolutions of Saturn	*upajāti*
66-70	Revolutions of Saturn in signs to thirds	*indravajrā*
71	Revolutions of Saturn in fourths	*upajāti*
72	Common divisors of a_{sa} and b	*anuṣṭubh*
73-76	Revolutions of the Moon's Anomaly up to degrees	*indravajrā*

77	Revolutions of the Moon's Anomaly in minutes	*upajāti*
78	Revolutions of the Moon's Anomaly in seconds	*indravajrā*
79	Revolutions of the Moon's Anomaly in thirds	*upajāti*
80	Revolutions of the Moon's Anomaly in fourths	*indravajrā*
81	Intercalary days since the beginning of the *yuga*	*anuṣṭubh*
82	Omitted days	*anuṣṭubh*
83	Sun's declination	*anuṣṭubh*
84	Intercalary Months since the beginning of the *yuga*	*upajāti*
85-88	Intercalary months in signs, degrees and minutes	*indravajrā*
89-90	Intercalary months in seconds, thirds and fourths	*upajāti*
91	Common divisors of a_{am} and b_{am}	*anuṣṭubh*

verses of the set do not concern directly planetary matter. The three first verses of this set (verses 81-83) are like loose strings in this regular ordered structure. The 91 verses open with a short introductive sentence:

> एते ग्रगकुट्टाकाराः श्लोकैरप्युपनिबाध्यन्ते।
>
> These planet pulverisers (*graha-kuṭṭākara*) are indeed composed in verses (*śloka*).

The term used to indicate the composition of the text is the verb *upa-ni-bandh-*. The sentence here most certainly contrasts here the prose commentary to this versified part of Bhāskara's text. But we can also note that such a verbal form, indicates a bounding together (*bandh-*), with technical prefixes. The act of composition, if we read it etymologically, can be understood as requiring the grouping of heterogenous elements, into a homogenized compact form.

In several verses the fact that the numbers have been learned by heart, and are then remembered (*smṛti*) is evoked. Therefore, one can always argue that surprises in a regular form may help the data to be impressed in the mind[29]. However, since we have no testimony concerning the use of this data, this hypothesis remains very speculative.

The author of these verses gives some indication that the format used is specific. Thus, in verse 9, the numbers are considered to be laid down (*vyavasthita*) almost as when they are noted on a working surface before being computed on.

It is difficult then to comment on the non regularity of the meter in the beginning and in the end of the set. Is this the sign that these verses were compiled? Were such variations used as mnemonic devices to remember the data? Similarly, it is difficult to decide whether this set of verses provide an example of syntactical table. If its structure is quite regular, it isn't absolutely so (a certain number of verses don't exactly follow the given pattern). However, one can further note that these verses, like other texts stocking numbers, are not quite exactly formulated in natural language. Thus the layered orders in which the numbers are stated, the use of versification, and the technical language, all concur to make of such a text, a very specific object that shares some properties with tabular displays.

7.4 A side effect: how quantities are expressed

The long successions of numerical compounds in verses, gives rises to peculiar expressions concerning quantities and numbers. Verses clearly state, such as in verse 12 (*gaṇitena labhyate*), that the numbers are obtained through computation. In most cases the numbers are given in *dvandva* compounds, using regular number names and word-numerals (*bhktasaṅkhyā*)[30]. Appendix C lists the names used for each number[31]. Full numbers are sometimes stated as well. More generally, the enumeration is not restricted to numbers smaller than nine. In many cases, as this is standard with such expressions, the number compounds are in a plural form, as if they consisted in a heap (*nicaya, nivaha, samūha*) of numbers, and particularly of digits who noted in due order with the decimal place-value notation would read as a higher number. This is sometimes made explicit, as in the much quoted verse 9:

> लब्धराशिनिचयः व्यवस्थितः पुष्कराग्निकृतनागबाहवः॥९॥
>
> A heap which is a quotient quantity (*labdharāśinicaya*) is laid out as <the following numbers> three-three-four-eight-two [28 433]||9||

The reference to a quantity <made of (the numbers)>, is the code I have used in the translation, to indicate that the compound listing the units was in the plural form while the quantity (*rāśi*) it refers to is in singular[32]. The term used for quantity, *rāśi*, has literally the meaning of "heap'', so that the expression *labdha-rāśi-nicaya* used above can seem redundant. Indeed, the term translated as "heap'' here, is also sometimes used as a synonym for quantity, as in the subsequent verse 10:

> तत्रलब्धनिचयः विकथयतेरुद्रवह्निरसनन्दपन्नगाः ॥१०॥
>
> Here the quotient quantity (*labdhanicaya*) is separately announced as eleven-three-six-nine-eight [896 311]||10||

In the next verse again, the term is used in a plural compound maybe as representing the digits as "heaps'':

> भूतबाणशरचन्द्रकुञ्जराः सागराम्बुनिवहाश्च लब्धकः ॥११॥
>
> The quotient is <made of the numbers> five-five-five-one-eight and heaps (*nivaha*) <that are > four-four [4 481 555]||11||

A same verse, can contain unit compounds in the singular and in the plural, as in verse 4:

> लैप्तोऽद्रिवेदरसरूपमतंगगजो ऽधो रूपांकदस्रभुजगद्विकनागवेदाः॥४॥
>
> For minutes seven-four-six-one-eight [81647], then one-nine-two-eight-two-eight-four [4 828 291]||4||

In verse 10 and subsequently others, several compounds are used for a same number

> राशितो ऽपि रसदस्रतापसा व्योमवेदगगनाशविनो गुणःल
>
> For signs also the multiplier is six-two-seven zero-four-zero-two [2 040 726] ||10ab||

More generally the link between the quantity designated and its stated amount is linked to the operation of measuring (*sammita, pramāna*[33]).

Thus, for instance in verse 51

> नन्दाद्रिवस्वष्टनभो ऽद्रिवेदा राशिर्गुणाख्यः खलु भागजातः।
> भूतांकतिग्मांशुनवाग्नितुल्यं लाभप्रमाणं प्रवदन्ति तजज्ञाः॥५१॥
>
> Indeed what is called a multiplier produced for degrees is the quantity nine-seven-eight-eight-zero-seven-four [4 708 879]|
>
> Those who know indicate that the measure of the quotient is equal to five-nine-twelve-nine-three [391 295]||51||

To sum it up, multipliers and quotients are quantities (*rāśi* or any word meaning heap- note that *rāśi* is also word for "signs'') to which a number *(saṅkhyā*) or a measure is associated. Thus, in verse 63

> प्रत्पराणां गुणकारराशिः अंगक्षामभूगगणांकसंख्यः।
>
> For fourths the multiplier quantity has the number six-seven-zero-nine [9 076]|

Although the convolution, shortness or spinning of these phrases rests certainly on meter requirement, they nonetheless open a door on the differences that could have existed in between numbers, measures and quantities, for Bhāskara. This underlying reflexion on the links between quantity, number and measure may be one door to the intended aim of these verses.

8. Conclusion

To end this study, let us come back to the questions raised in the introduction: Why and how then were these solutions collected? Who could have used them, and why did they not have any posterity? Of course answers to why such verses had no posterity in Sanskrit astral and mathematical literature are bound to be very speculative.

To attempt an answer, we would need to know for what kind of computations were more general pulverizers used: Were they "practical'' tools for making almanacs and calendars (*pañcāṅga*)? Were they mathematicians' games?

As we have seen, it is difficult indeed to decide if such verses belong more to *gaṇita* than to *jyotiṣa*. Such questions also bring us to the place were mathematics and astral science meet in astral literature: precisely were procedures are ascertained, grounded, and playfully explored.The numbers undertaken here being measured with units in thirds and fourths, evaluated with the "theoretical'' epoch of the beginning of the great *yuga*, and found in the commentary of a *siddhānta* indeed suggest that such couple solutions were not tabulated to make a practical calendar but rather had more fanciful aims: explorations, verifications, fun? The puzzle of these verses, can thus be seen as highlighting a key element of the historical study of numerical tables: it is by observing not only how they were derived but how they were used, that we can unravel more properly their context and function. Reversely then, with no information on the use, we are left with but a handful of questions.

Abbreviations and Notations

Ab The *Āryabhaṭīya* of Āryabhaṭa (Shukla 1976).

BAB The *Āryabhaṭīyabhāṣya* of Bhāskara (Shukla 1976) (Keller 2006).

BAB.2 corresponds to the *Āryabhaṭīyabhāṣya*'s commentary on the second chapter of the *Āryabhaṭīya*, the one devoted to mathematics. Thus BAB.2.32 corresponds to Bhāskara's commentary on verse 32 of the second chapter of the *Āryabhaṭīya*.

BAB.Ku corresponds to the portion of *Āryabhaṭīyabhāṣya* containing the 91 verses studied here. Thus BAB.Ku.21 corresponds to verse 21 of this series.

Conventions

I've used the usual transliteration of Sanskrit, when I wanted to make words and phrases accessible to non-sanskritists. In general then, when quotations are made for the benefit of sanskritists alone, I've used devanagari transcriptions.

Mathematical Notation

KU(a, b, c)[x,y]

Pulverizer (*kuṭṭka* or *kuṭṭākara*) for $a,b,x,y \in \mathbf{N}^*$, $c \in \mathbf{Z}^*$, a, b, c known, x,y unknown.

a notation for the problem and procedure to solve:

(α, β) notes a specific solution of this equation. With an algorithmic vocabulary: the input is (a,b,c), the output are all given values of (α, β). In the verses studied here (a,b) are implicit, c=-1 and (α, β) is stated using standard names for sanskrit numbers and word numerals.

Appendix A: Translation

This section presents a translation of the verses that end Bhāskara's commentary of the mathematical part of the *Āryabhaṭīya*. Shukla's 1976 edition of the Sanskrit text, which is not reproduced here, was used for this purpose.

In this translation, [] stand for addendums by the editor, K. S. Shukla, <> stand for my own addendums, () give specifications.

[Result of a *kuṭṭākāra* (pulverizer) with one as decrease]

These planet pulverizers (*graha-kuṭṭākāra*) are indeed composed in verses (*śloka*). It is as follows:

BAB.Ku.1 Hommage to Śiva (as Śaṃbhu, "the benevolent'') cause of the destruction and creation of the world, who is embodied in the <asterisms> starting with the sun (*Bhāskara*), whose vital spirit is ten thousand <times that> of the sun.||1||

BAB.Ku.2 I indicate here with delight for those who know them the pulverizer (*kuṭṭākara*) and quotient (*lābha*) <derived> two by two from revolutions and so forth in due order ||2||

[For the Sun]

BAB.Ku.3 For the sun, [two-zero-six-four-nine] [9 4602]. The quotient for it derived from the revolution <number> (*bhagaṇa*) is <made of the numbers> nine-five-two [259][34]. And for signs (*rāśi*), <the multiplier is made of the numbers> eight-seven-zero, three and eleven [113 078]

The quotient should be the numbers five-one-seven-three [3 715]||3||[35]

BAB.Ku.4 Degrees of <the multiplier> equal to three-seven-eight-nine-five [59 873] the quotient is taught as <made of the numbers> [eleven-zero-nine-fives] [59011]|[36]. For minutes seven-four-six-one-eight [81647], and then one-nine-two-eight-two-eight-four [4 828 291][37]||4||

BAB.Ku.5 [For seconds], the multiplier (*guṇakāra*) [is indeed produced as <the numbers> one-seven-nine-zero-six [60 971], and then here the quotient has the number one-nine-fourteen-five-three-three-six-one two [21 633 5491]. And after that, for thirds||5||

BAB.Ku.6 Here two-eight-[seven-four-four-one 144 782] are wished as multiplier. And subsequently (*adhas*) one-nine-four-one-eight-six-two-two-eight-zero-three [30 822 671 491]||6||

BAB.Ku.7 The multiplier quantity (*guṇakrarāśi*) [produced] for fourths should be known as a number equal to twenty-six-four-nine [9 4 26]. Then here the quotient <is made of the numbers> one-nine-four-nine-one-two-zero-four-zero-twelve [1 20 402 191491]||7||

BAB.Ku.8 Just one is the remembered denominator for the sun, for revolutions and so forth until fourths, zero-zero-five-seven [7500] <is a common divisor of revolutions and civil days>[38].||8||

[For the Moon]

BAB.Ku.9 When pulverized (*prakuṭṭita*) [by the revolutions] of the moon, <the mulitplier is made of the numbers> seven-three-eight-six-seven-seven [776 837]|

A heap which is a quotient quantity (*labdharāśinicaya*) is laid out (*vyavasthita*) as <the following numbers> three-three-four-eight-two [28 433]||9||

BAB.Ku.10 For signs also the multiplier is <made of the numbers> six-two-seven zero-four-zero-two [2 040 726]|. Here the quotient quantity (*labdhanicaya*) is separately announced as eleven-three-six-nine-eight [896 311]||10||

BAB.Ku.11 What is called the multiplier for the remaining degrees is < made of the numbers> one-twelve-zero-four-three [340 121]|. The quotient is <made of the numbers> five-five-five-one-eight and heaps (*nivaha*) <that are> four -four [4 481 555]||11||

BAB.Ku.12 In minutes, <the multiplier> is measured (*sammita*) as eight-five-eleven-two [21 158]|. Nine-eleven-seven two-seven [sixteen 16 727 119][39] is obtained through computation ||12||

BAB.Ku.13 <The multipler> produced with the moon's seconds is measured as four-thirteen-six-one [16 134]|. The next is remembered as <made of the digits> five-one-three-fourteen-three five-six-seven [765 314 315]||13||

BAB.Ku.14 In thirds, they point out eighteen-five-six [6 518] as a number for multiplier.| The following (i.e. the quotient) is <made of the digits> twenty seven-nine-one-three-eight-zero-five-five-eighteen [18 550 831 927]||14||

BAB.Ku.15 In what follows thirds (e.g fourths) three-nine-two-two [2 293] are mentioned in due order|. In due order, eleven-five-seven-two-four five-six-fifteen-nine-three [3 91 565 427 511]||15||

BAB.Ku.16 Starting with the moon's degrees the divisor should be set here in due order as five [5] five [5] after five [5] with thirty-two-seven [732]||16||

[For the Moon's Apogee]

BAB.Ku.17 The number equal (*samāna*) to nine-seven-eight/seven-seven/eight-six six-eight-one [718 667 879]| is [here the multiplier] of the moon's apogee (*indūcca*) which is shown with all the asterisms (*bhacakradṛṣṣṭa*) and the quotient is one-six-three-two-two-two [222 361]||17||

BAB.Ku.18 The multiplier quantity measures (*pramāṇa*) four-five- [two-two-zero two-one six [61 202 254]| Here in due order for signs the computed quotient is seven-three-two-seven-two-two [227 237]||18||

BAB.Ku. 19 <The multiplier> is stated in due order as the numbers four-eight-five-six-six and nine-eight-one [18 966 584]|. In due order for degrees, with a computation the quotient is <made of (the numbers)> three-two-six-twelve-one-two [2 112 623]||19||

BAB.Ku. 20 The multiplier quantity obtained here for minutes is thirty-two-zero-seven-eight-zero-five [5 087 032]|. The quotient is here told to consist of the numbers twenty-seven-six-seven-nine-nine-three-three [33 997 627]||20||

BAB.Ku.21 The multiplier number for seconds should be one-seven-thirty-two-seven [73 271]|

Those who know it wish the quotient to be three-six-ten-eight-three-nine accompanied by two [29 381 063]||21||

BAB.Ku.22 What indeed is here derived from thirds is equal in numbers to eight-six-seven-three-nine [93 768]|. And the quotient is five-nine-three-one-one-zero-six-five-two-two [2 256 011 395]||22||

BAB.Ku.23 In due order, what is [derived from fourths is equal in numbers to four-seven-eight-nine-six-one [169 874]|. And the quotient is eleven-eight-one-three-zero-five-two-two-five-four-two [245 225 031 811]||23||

BAB.Ku.24 For signs etc in due order [the divisors should be set as told here], that is,| Twelve [12] and then five [5], [five 5] and up to thirds||24||

[For the Moon's Node]

BAB.Ku.25 They say the multiplier consists of the number seven-seven-one-six-zero-six-five-two-six [625 606 177]|. [And the divisor of the node] concerning revolutions has rightly the measure two-seven-zero-two-nine [92 072]||25||

BAB.Ku.26 Those who know that declare one-zero-four-three-six-five-five-eleven [1 155 63 401] |

[Here the quotient] produced in in due order for signs is three-nine-zero-four-zero-two [204 093]||26||

BAB.Ku.27 One-seven-six-three-four-six-three-two [23 643 671] is for the multiplier|. Following it the quotient stated as numbers for degrees in due order is one-nine-six-two-five-twelve [1 252 691] ||27||

BAB.Ku.28 For minutes, the multiplier's numbers are declared to be eight-five-eight-one-six-one-four [4 161 858]|. They proclaim a quotient equal to <the numbers> five-three-two-zero-three-two-thirteen [13 230 235]||28||

BAB.Ku.29 With a reasoning (*yukti*) on the multiplier for seconds which is a quantity four-nine-seven-two-seven-eight [872 794]|. In due order here the quotient should be seven-zero-eight-two-seven-four-six-six-one [166 472 807]||29||

BAB.Ku.30 That which is a multiplier quantity in thirds is here eight-six-six-seven-three [37 668]|

The quotient also has the measure of seven-zero-five-seven-seven-zero-one-three-four [431 077 507]||30||

BAB.Ku.31 They say that the multiplicand for fourths consists of the numbers nine-three-nine-eight-six-one [168 939]| And the quotient is one-seven-five-seven-nine-five-one-zero-zero-sixteen and one [116 001 597 571]||31||

BAB.Ku.32 The divisors for revolutions is stated to be two, for the signs, twelve| For the assemblage of degrees, etc five is stated by the wise ones||32||

[For Mars]

BAB.Ku.33 For the revolutions of Mars, the produced multiplier should be thirteen-twelve-ten-sixteen [16 101 213]|. Here the quotient is in due order told to measure seven-three-four-three-two [23 437]||33||

BAB.Ku.34 In that case, the measure of the multiplier is nine-four-zero-five-one-two-four-three [34 215 049]| For signs in due order, the amount of the quotient is specified to be three-four-six-seven-nine-five [597 643]||34||

BAB.Ku.35 They say that the multiplier for measures in degrees is four-zero-four-nine-one-thirteen [1 319 404]|. Here the number of the quotient is determined as equal to one-nine-three-one-nine-six [691 391]||35||

BAB.Ku.36 The multiplier [for minutes] is two-nine-one-three-six-eight-one [1 863 192] according to those who know multipliers|. Here the quotient is seen to be equal to one-three-[seven-zero-eight-five]-eight-five [58 580 731]||36||

BAB.Ku.37 For seconds, the multiplier is recognized as <made of (the numbers)> six-six-two-five-five-ones [155 266]. The quotient, they say is equal to five-five-six-three-zero-nine-two-nine-two [292 903 655]||37||

BAB.Ku.38 In that case, for a number in thirds, in due order, the multiplicand measures three-six-two-eight-eighteen [188 263]. And the quotient is <made of (the numbers)> five-three-six-seven-seven-zero-nine-zero-three-one-two [21 309 077 635]||38||

BAB.Ku.39 For fourths the multiplier's amount measures two-two-seven-eight-four [48 722]

And the quotient is <made of (the numbers)> one-five-six-three-zero-two-four-eight-eight-zero-three-three [330 884 203 651]||39||

BAB.Ku.40 For revolutions (*maṇḍala*) and signs (*gṛha*) the memorized divisor (*cheda*) is twelve. For others beginning with degrees five after five. This is a rule (*sthiti*)||40||

[For the *śīghrocca* of Mercury]

BAB.Ku.41 The multiplicand [for mercury] consists in the [numbers] six-seven-two-[seven-eight-five-three 23 587 276]|. For revolutions the quotient in due order is the number nine-twelve-eight-six-two [268 129]||41||

BAB.Ku.42 With signs in due order the multiplier should be equal to [nine]-one-eight-six-nine-eight-five [5 896 819]|. And the quotient <obtained> according to the rule (*vidhi*) consists of the numbers indicated as seven-eight-three-four-zero-eight [804 387]||42||

BAB.Ku.43 In degrees, they say that the multiplier is four-two-four-nine-five-eight-one [1 859 424]|. There the quotient indeed measures the number nine-three-three-nine-zero-six-seven [7 609 339]||43||

BAB.Ku.44 They say that the established multiplier for minutes is two-five-nine-four-five-one [154 952]|. The quotient whose number is five-nine-six-six-four-zero-eight-three [38 046 695] is specified by mathematicians||44||

BAB.Ku.45 The quantity called the multiplier for seconds measures two-seven-one-three-fifteen [153 172]|. Here, the quotient <obtained>

according to the rule (*viddhi*) is nine-seven-one-eight-seven-five-six-five-two-two [2 256 578 179]||45||

BAB.Ku.46 The multiplier as a number for thirds has been told here: they say that it is zero-nine-eighteen [184 890]|. And the quotient is equal to one-nine-zero-five-six-four-one-three-four-three-sixteen [163 431 465 091]||46||

BAB.Ku.47 For fourths the multiplier is six-seven-two-eight-zero and one [108 276]. Now the quotient,| one-nine-eight-five-six-one-two-six-five-two-four-seven-fives [5 742 562 165 891]||47||

BAB.Ku.48 Twenty [20] and then sixty [60] is the divisor produced from revolutions and signs|. Wise ones indicate for degrees etc. <except for thirds and fourths> in due order five [5]||48||

[For Jupiter]

BAB.Ku.49 The assemblage <of numbers> produced for multiplier is eight three-zero-three-five-zero-six-seven [76 053 038]|. The quotient for Jupiter, in due order for revolutions, will be five-five-five-seven-one [17 555]||49||

BAB.Ku.50 The multiplier seen with a method for signs is four-seven-two-three-five-two-eight-two [28 253 274]|. The quotient according to an established computation is said to be in this case the measure nine-twenty-five-eight-seven [78 259]||50||

BAB.Ku.51 Indeed what is called a multiplier produced for degrees is the quantity nine-seven-eight-eight-zero-seven-four [4 708 879]|. Those who know indicate that the measure of the quotient is equal to five-nine-twelve-nine-three [391 295]||51||

BAB.Ku.52. Those who know wish the multiplicand in due order for minutes to be <made of (the numbers)> seven-one-seven-zero-three-eight [830 717]|. The quotient for minutes, having computed <it> is told to be nine-one-eighteen-four-one-four [4 141 819]||52||

BAB.Ku.53 For seconds, the category of multiplier is by means of <the numbers> one-six-eight-two-eight-six [682 861]|. In that case, <the numbers> one-nine-nine-seven-seven-two-four-zero-two [204 277 991] are said to be the quotient||53||

BAB.Ku.54 And for thirds here indeed the number thirty-two-six-nine-seven-one [179 632] is the multiplier|. Seven-zero-seven-sixteen-two-four-twos connected with thirty-two [3 224 216 707] are told to be the quotient||54||

BAB.Ku.55 Here the multiplier for fourths is measured as one-three-three-five-eighteen [185 331]. The quotient is| one-seven-one-seven-eight-four-zero-nine-five-nine-nine-one [199 590 487 171]||55||

BAB.Ku.56 For revolutions and signs the divisor is remembered as twelve|. The divisors of the quantities (*rāśi*) in degrees, minutes etc are told to be precisely five ||56||

[For the *śīghrocca* of Venus]

BAB.Ku.57 For revolutions, the multiplier for the apogee of Venus is nine-four-zero-six-four-zero-seven [70 046 049]|. The quotient is said to be equal to four-three-seven-eleven-three [311 734] as a number <obtained> in due order according to a rule for numbers||57||

BAB.Ku.58 The multiplier for signs is pointed out to be two-five-four-zero-one-seven-eight-threes [38 710 452]|. The quotient is an amount equal to one-three-three-seven-six-zero-two [2 067 331]||58||.

BAB.Ku.59 The multiplicand for degrees is equal to [the number] two-four-seven-one [five-four six 6 451 742]|. Then here the quotient is the quantity five-five-six-six-three-three-zero-one [10 336 655]||59||

BAB.Ku.60 For minutes the category of multiplier is the quantity six-six-two-four-one-fourteen[1 414 266]|. Here the quotient is equal to [one-three-nine-one-five-nine-five-three-one [135 951 931]||60||

BAB.Ku.61 In seconds they say that the result called "multiplier" is equal to eight-two-eight-three-four-six [643 828]|. And the quotient should be one-seven-twelve-three-four-three-one-seven-three [3 713 431 271]||61||

BAB.Ku.62 In due numerical order the multiplicand which is a quantity for thirds is the number two-eight-seven-three-twelve [123 782]|. Here the quotient is equal to three-eight-two-nine-two-five-six-three-eight-two-four [42 836 529 283]||62||

BAB.Ku.63 For fourths the multiplier quantity has the number six-seven-zero-nine [9 076]|

And the quotient is one-five-eight-six-six-seven-two-five-four-eight and eighteen [188 452 766 851]||63||

BAB.Ku.64 For revolutions together with signs the divisor is remembered as twelve|. And five [5] just five [5] is set for the degrees and so forth for Venus||64||

[For Saturn]

BAB.Ku.65 In revolutions, the multiplier is indicated by those who know to be eleven-two-five-six-zero-zero-three-eleven [113 065 211]|. The quotient quantity of the son of the sun (Saturn) is two-zero-five-zero-one [10 502]||65||

BAB.Ku.66 In this case, the multiplier quantity is four-eight-five-nine-three-one-one-six [61 139 584]|. But for signs in due order the number seven-four-one-eight-six [68 147] is declared equal to the quotient||66||

BAB.Ku.67 What is called multiplier arises from nine-three-one-six-five-nine-eight-one [18 956 139]|. For up to degrees (*aṃṣāvadhi*) the quantity (*samūha*) produced for quotient is three-six-eight-three-three-six [633 863]||67||

BAB.Ku.68 The quantity (*samūha*) produced as the multiplier is seen to be seven-four-seven-four-six-two [264 747]|. And in due order for minutes here, being computed, the quotient measures three-six-eleven-three-five [531 163]||68||

BAB.Ku.69 The quantity one-two-zero-eleven-eight [811 021] is produced for seconds for the multiplier|. Here the quotient is set to be of the measure of seven-eight-two-nine-two-six-seven-nine [97 629 287]||69||

BAB.Ku.70 The multiplicand for amounts of thirds is indicated as twelve-three-zero-nine[-one 190 312]|. The quotient is seven-zero-five-[five-six]-five-four-seven-three-ones [1 374 565 507]||70||

BAB.Ku.71 For fourths, the multiplier is nine-zero-five-five-eight- one [185 509]. Now the quotient is| eleven-two-four-nine-four-two-nine-three-zero joined with eight [80 392 494 211]||71||

BAB.Ku.72 The divisor of revolutions is four [4], and just twelve [12] for signs| In due order the exact divisor from degrees (*lava*) is said to be five for Saturn (<son> of the sun) ||72||

[For the Moon's Anomaly]

BAB.Ku.73 The multiplicand produced for revolution is three-five-nine-three-one-twelve-four-threes [341 213 953]|. And the quotient for the moon's anomaly is three-nine-one-three-eight-three-twelve [12 383 193]||73||

BAB.Ku.74 The multiplier for its signs should be equal to three-zero-seven-seven-two-two-eight-seven [78 227 703]|. The quotient is proclaimed by mathematicians to be two-eight-zero-eight-six-zero-four- three [34 068 082]||74||

BAB.Ku.75 The multiplier born from the degrees should amount to three-six-two-seven-eight-one-six-two [26 187 263]|. They proclaim that the quotient amounts to one-six-seven-five-three-one-two-four-three [342 135 761]||75||

BAB.Ku.76 In this <case>, the multiplier quantity produced from minutes is four-two-eight-three-seven-three-four [4 373 824]|. Indeed there the quotient is five-one-three-two-three[-six-eight-two-four]-three [3 428 632 315]||76||

BAB.Ku.77 The multiplier quantity for seconds is equal to seven-three-eight-three-one [13 837|. Here the quotient is equal to nine-seven-eight-seven-zero-eight-zero-five-six [650 807 879]||77||

BAB.Ku.78 The multiplier obtained for thirds is seen to have as a measure zero-eight-eight-three-twelve [123 880|.The quotient is a quantity in thirds is eleven-zero-seven-six-four-three-nine-five-nine-four with three [349 593 467 011]||78||

BAB.Ku.79 Here the multiplier for fourths is twenty-four-three-two-fourteen [142 324], but the quotient quantity is| one-five-two-nine-eight-seven-two-eight-five-eight-nine-zero-twenty-four [24 098 582 789 251]||79||

BAB.Ku.80 The reducer which is a denominator should be known for signs to be twelve [12]|

In due order for degrees and so forth the denominators are seen in due order by the enlightened ones to be five [5]||80||

[For Intercalary Days]

BAB.Ku.81 The <number of> intercalary <days> which is a multiplier is equal to nine-zero-five-two-seven-six [672 509]|. The quotient quantity in due order is zero-[seven-six]-zero-two [20 670]||81||

[For Omitted Days]

BAB.Ku.82 The <number of> omitted <days> which is a multiplier is one-three-zero-seven-four-two-six[6 247 031]|. The quotient also is mentioned as nine-four-seven-seven-nine [97 749]||82||

[For the Sun's Declination]

BAB.Ku.83 A multiplier in due order for the declination is seven-four-three [347|

In due order, the quotient quantity is seen to be [one-four]-one [141]||83||

[For Intercalary Months]

BAB.Ku.84 For the intercalary months in a *yuga*, the multiplicand is seen as seven-one-three-zero-zero-nine and eighteen [18 900 317]|. Here also, in this case, the quotient for revolutions is constantly five-eight-zero-nine-one [19 085]||84||

BAB.Ku.85 The quantity which is a multiplier is equated with <the numbers> one-nine-three-zero-eleven-two-twelve [122 110 391]|. Here the quotient quantity produced from signs consist of the numbers three-four-six-nine-seven-fourteen [1 479 643]||85||

BAB.Ku.86 The multiplier quantity should be six-three-eight-four-three-seven-four-two [24 734 836]|. In due order the quotient established with a computation of degrees is one-five-five-one-nine-nine-eight [8 991 551]||86||

BAB.Ku.87 In minutes, the multiplier quantity is seen to consist of the numbers eight-seven-four-fourteen-eight-three [3 814 478]|. They say that the quotient has a measure of nine-one-eight-seven-nine-one-three-eight [83 197 819]||87||

BAB.Ku.88 The established multiplier for seconds is seen to measure nine-four-five-two- fourteen [142 549]|. They say that the quotient is eleven-seven-forty-eight-five-six-eight-one [186 548 711]||88||

BAB.Ku.89 And also for thirds the number [for the multiplier] is six, zero-six-four-three-one [134 606]|. And the quotient is three-eight-six-nine-three-two-nine-six-five with ten [10 569 239 683]||89||

BAB.Ku.90 For fourths the quantity obtained for multiplier is declared to be twelve-four-three-nine [93 412]|. Here the quotient is equal to one-three-five-eight three-sixteen-eight-zero-zero-four-four [440 081 638 531]||90||

BAB.Ku.91 The reducer for revolutions with signs is exactly twelve [12]| For the remaining, when reducing the divisor is five [5] and also five [5]||91||

Thus ends the mathematical quarter in Bhāskara's work, a commentary on the *Āryabhaṭatantra*.

Appendix B: Tabular data in the verses

Tabular displays of the values provided in these verses have already been published by K. S. Shukla[40]. Here, I would like to enhance his tables by explaining and commenting on the values found. In all these tables, for resolutions of KU(*a*, *b*, -1)[*x*,*y*], the columns for *a* and *b* are implicit. To mark their implicit status they are tabulated in italic. They use data more or less elucidated by Bhāskara or Āryabhaṭa in other parts of the text, which are referenced when they have been traced. Although the values for *a* and *b* are not specified in the verses, each section of versified data, ends with a list of divisors used to reduce simultaneously *a* and *b*. These divisors are listed in a separate column, labeled *d*. A given value of *d* reduces the values of *a* and *b* derived from the previous row. For instance, the value of *d* for seconds in Table 7, 5, divides the already reduced values for *a* and *b* used for minutes (68 167 872, 86 225); the value for *a* is converted to seconds by a multiplication by 60 (4 090 072 320). The couple (4 090 072 320, 86 225), when divided by 5 yields (818 014 464, 17 245), the values for *a* and *b* noted in the row for seconds. The divisor for revolutions divides the original values of *a* and *b*.

I've recomputed all the solutions of these pulverizers. I've done this using spreadsheets, that enable one to see and check all the intermediary values computed in a pulverizer process. These spreadsheets which actually enable one to compute any pulverizer problem are available at http://halshs.archives-ouvertes.fr/halshs-00760116. I am ready to improve their readability on a simple electronic request. In most cases the tabulated solutions are the smallest possible. There is one notable exception, noted in bold. A rectified value of a manifest scribal error has in the same way been underlined.

Standard Values

In most cases KU(a_g,b,-1)[x,y_g] is the data coined implicitly and explicitly in a given verse, were:

a_g is the revolution number of planet g in a great *yuga*, as given in Ab.1.3-4.

b is the number of civil days in a great *yuga* (1 577 917 500).

y_g is the number of revolutions of g since the beginning of the great *yuga*.

x is the number of civil days elapsed since the beginning of the great *yuga*.

Sun

The tables here, corresponds to the data stored in verses 3-8.

KU(a_s,b,1)[x,y_s] are considered. a_s is the number of revolutions of the sun in the great *yuga*, 4 320 000. BAB.2.32-33 ex 7, also underlines that

Table 5: Tabulated values for the Sun

Measuring Units	*d*	a_s	*b in days*	*x*	*y*
Revolutions	7500	*576*	*210 389*	94 602	259
Signs	1	*6 912*	*210 389*	113 078	3 715
Degrees	1	*207 360*	*210 389*	59 873	59 011
Minutes	1	*12 441 600*	*210 389*	81 647	4 828 291
Seconds	1	*746 496 000*	*210 389*	60 971	216 335 491
Thirds	1	*4 478 976 000*	*210 389*	144 782	30 822 671 491
Fourths	1	*2 687 385 600 000*	*210 389*	9 426	120 402 191 491

Table 6: Tabulated values for the Moon

Measuring Units	*d*	a_c	*b in days*	*x*	*y*
Revolutions	732	78 898	2 155 625	776 837	28 433
Signs	1	946 776	2 155 625	2 040 726	896 311
Degrees	5	5 680 656	431 125	340 121	4 481 555
Minutes	5	68 167 872	86 225	21 158	16 727 119
Seconds	5	818 014 464	17 245	16 134	765 314 315
Thirds	5	9 816 173 568	3449	**6 518**	**18 550 831 927**
Fourths	1	588 970 414 080	3449	2 293	3 91 565 427 511

Table 7: Tabulated values for the Moon's Apogee

Measuring Units	*d*	a_{cu}	*b in days*	*x*	*y*
Revolutions	1	*488 219*	*1 577 917 500*	718 667 879	222 361
Signs	12	*488 219*	*131 493 125*	61 202 254	227 237
Degrees	5	*2 929 314*	*26 298 625*	18 966 584	2 112 623
Minutes	5	*35 151 768*	*5 259 725*	5 087 032	33 997 627
Seconds	5	*421 821 216*	*1 051 945*	73 271	29 381 063
Thirds	5	*5 061 854 592*	*210 389*	93 768	2 256 011 395
Fourths	1	*303 711 275 520*	*210 389*	169 874	245 225 031 811

7500 is a common divisor of a_s and b, and works with the reduced values: a_s/b=4 320 000/1 577 917 500= 576/210 389

Moon

The tables here, corresponds to the data stored in verses 9-16.

KU(a_c,b,1)[x,y_c] is considered, where, a_c is the number of revolutions of the Moon in the great *yuga*, 57 753 336.

In one instance, the values for thirds, the smallest solution KU(a,b,-1)[α,β] is not stated, but KU(a,b,-1)[$\alpha+b,\beta+a$]. Further, Shukla has corrected for minutes faulty readings from manuscripts, as underlined in the translations of these texts.

Moon's Apogee

The tables here, corresponds to the data stored in verses 17-24. KU(a_{cu},b,1)[x,y_{cu}] is considered. a_{cu} is the number of revolutions of the Moon's Apogee in the great *yuga*, 488 219.

Moon's Ascending Node

Verses 25-32 tabulate solutions of KU(a_p, b, -1)[x, y_p],

where a_p is the number of revolutions of the moon's node in an opposite direction (i.e. ascending, *pātaviloma*) in a *yuga* (Ab.1.4 gives 232 226 revolutions).

Mars

Verses 33-40 tabulate solutions of KU(a_b, b, -1)[x, y_b],

where a_b is number of revolutions of Mars (*bhaumya*) in a *yuga*. According to Ab.1.3, a_b=2 296 824 revolutions.

(*Śīghrocca* of) Mercury

Verses 41-48 tabulate solutions of KU(a_{bu}, b, -1)[x, y_{bu}].

a_u is the number of revolutions of (the *śīghrocca* of) Mercury (*bhudha*) in a *yuga*. According to Ab.1.3 a_{bu}=17 937 020 revolutions[41].

Table 8: Tabulated values for the Moon's Ascending Node

Measuring Units	***d***	a_p	***b in days***	***x***	***y***
Revolutions	2	*116 113*	*788 958 750*	625 606 177	92 072
Signs	12	*232 226*	*131 493 125*	1 155 63 401	204 093
Degrees	5	*1 393 356*	*26 298 625*	23 643 671	1 252 691
Minutes	5	*16 720 272*	*5 259 725*	4 161 858	13 230 235
Seconds	5	*200 643 26*	*1 051 945*	872 794	166 472 807
Thirds	5	*2 407 719 168*	*210 389*	37 668	431 077 507
Fourths	5	*144 463 150 080*	*210 389*	168 939	116 001 597 571

Table 9: Tabulated values for Mars

Measuring Units	***d***	a_b	***b in days***	***x***	***y***
Revolutions	12	*191 402*	*131 493 125*	16 101 213	23 437
Signs	1	*2 296 824*	*131 493 125*	34 215 049	597 643
Degrees	5	*137 80 944*	*26 298 625*	1 319 404	691 391
Minutes	5	*165 371 328*	*5 259 725*	1 863 192	58 580 731
Seconds	5	*1 984 455 936*	*1 051 945*	155 266	292 903 655
Thirds	5	*23 813 471 232*	*210 389*	188 263	21 309 077 635
Fourths	1	*1 428 808 273 920*	*210 389*	48 722	330 884 203 651

Table 10: Tabulated values for (the *śīghrocca* of) Mercury

Measuring Units	***d***	a_{bu}	***b in days***	***x***	***y***
Revolutions	20	*896 851*	*788 895 875*	23 587 276	268 129
Signs	30	*3 587 404*	*26 298 625*	5 896 819	804 387
Degrees	5	*21 524 424*	*5 259 725*	1 859 424	7 609 339
Minutes	5	*258 293 088*	*38 046 695*	154 952	38 046 695
Seconds	5	*3 099 517 056*	*210 389*	153 172	2 256 578 179
Thirds	1	*185 971 023 360*	*210 389*	184 890	163 431 465 091
Fourths	1	*1 428 808 273 920*	*210 389*	108 276	5 742 562 165 891

Table 11: Tabulated values for Jupiter

Measuring Units	***d***	a_{gu}	***b in days***	***x***	***y***
Revolutions	12	*30 352*	*131 493 125*	76 053 038	17 555
Signs	12	*364 224*	*131 493 125*	28 253 274	78 259
Degrees	5	*2 185 344*	*26 298 625*	4 708 879	391 295
Minutes	5	*26 224 128*	*5 259 725*	830 717	4 141 819
Seconds	5	*314 689 536*	*1 051 945*	682 861	204 277 991
Thirds	5	*3 776 274 432*	*210 389*	179 632	3 224 216 707
Fourths	5	*226 576 465 920*	*210 389*	185 331	199 590 487 171

Jupiter

Verses 49-56 tabulate solutions of KU(a_{gu}, b, -1)[x, y_{gu}],

where a_{gu} is the number of revolutions of Jupiter (*guru*) in a *yuga*. According to Ab.1.3, a_{gu} =364 224 revolutions.

(Śīghrocca of) Venus

Verses 57-64 tabulate solutions of KU(a_{bh}, b, -1)[x, y_{bh}],

where a_{bh} is the number of revolutions of (the *śīghrocca* of) Venus (*bhṛgu*) in a *yuga*. According to Ab.1.4, a_{bh} =7 022 388 revolutions.

Saturn

Verses 65-72 tabulate solutions of KU(a_{sa}, b, -1)[x, y_{sa}],

where a_{sa} is the number of revolutions of Saturn (*śani*) in a *yuga*. According to Ab.1.3 a_{sa} =146 564 revolutions.

Moon's Anomaly

Ab.3.4 defines the number of anomalistic revolutions of a planet during a great *yuga* as the difference between its revolution number and the revolution number of its apogee[42]. It thus provides a value for a_{ck} the number of anomalistic revolutions of the moon (*candrakendra*) during a great *yuga*. a_{ck} =57 753 336-488 219=57 265 117.

Verses 73-80 tabulate solutions of KU(a_{ck}, b, -1)[x, y_{ck}].

Intercalary days

(672 509, 20 670) is the smallest positive solution of y= (*66 389*x_{ad}*-1)/2 160 000*. But, how should one interpret astronomically the ratio 66 389/ 2 1 60 000?

In the *Mahābhāskarīya*, in MBh.1.22 the number of mean intercalary days in a mean solar year is, according to Shukla, assumed to be 11 + 389/6 000=66 389/6 000[43]. But how then should we adapt this knowledge to a planetary pulverizer?

Table 12: Tabulated values for (the *śīghrocca* of) Venus

Measuring Units	***d***	a_{bh}	***b in days***	***x***	***y***
Revolutions	12	585 199	131 493 125	70 046 049	311 734
Signs	12	7 022 388	131 493 125	38 710 452	2 067 331
Degrees	5	42 134 328	26 298 625	6 451 742	10 336 655
Minutes	5	505 611 936	5 259 725	1 414 266	135 951 931
Seconds	5	6 067 343 232	1 051 945	643 828	3 713 431 271
Thirds	5	72 808 118 784	*210 389*	123 782	42 836 529 283
Fourths	5	4 368 487 127 040	*210 389*	9 076	188 452 766 851

Table 13: Tabulated values for the revolutions of Saturn

Measuring Units	***d***	a_{sa}	***b in days***	***x***	***y***
Revolutions	4	*36 641*	*394 479 375*	113 065 211	10 502
Signs	3	*146 564*	*131 493 125*	61 139 584	68 147
Degrees	5	*879 384*	*26 298 625*	18 956 139	633 863
Minutes	5	*10 552 608*	*5 259 725*	264 747	531 163
Seconds	5	*126 631 296*	*1 051 945*	811 021	97 629 287
Thirds	5	*1 519 575 552*	*210 389*	190 312	1 374 565 507
Fourths	5	*91 174 533 120*	*210 389*	185 509	80 392 494 211

Table 14: Tabulated Values for the Moon's Anomaly

Measuring Units	***d***	a_{sa}	***b in days***	***x***	***y***
Revolutions	1	*57 265 117*	*1 577 917 500*	341 213 953	12 383 193
Signs	12	*57 265 117*	*131 493 125*	78 227 703	34 068 082
Degrees	5	*343 590 702*	*26 298 625*	26 187 263	342 135 761
Minutes	5	*41 23 088 424*	*5 259 725*	4 373 824	3 428 632 315
Seconds	5	*49 477 061 088*	*1 051 945*	13 837	650 807 879
Thirds	5	*593 724 733 056*	*210 389*	123 880	349 593 467 011
Fourths	5	*35 623 483 983 360*	*210 389*	142 324	24 098 582 789 251

This is a hypothetical construction in which verse 81 would compute KU(a_{ad}, b_{ad},-1)[x_{ad},y_{ad}], where a_{ad} is the mean number of intercalary days elapsed, 66 389/6 000 and b_{ad} would be the mean number of lunar days in a mean solar year, 360. The assumption would be that $a_{ad} / b_{ad} = Y_{ad} / x_{ad}$, where x_{ad} is the number of mean days elapsed since the beginning of a great *yuga*, and Y_{ad} the number of mean intercalary days elapsed.

Only a fractional part of Y_{ad} would be known: λ_{ad}, $Y_{ad}=y_{ad}+\lambda_{ad}$. The remaining part, y_{ad}, is unknown.

In this case $\lambda_{ad}=1/b_{ad}=1/360$. The interpretation of the result would be that when 1/360th of a lunar day is seen to be elapsed, then we know that 672 509 mean lunar days have elapsed since the beginning of the kali *yuga* and that the integral number 20 670 and 1/360th mean lunar days have been intercalated.

Omitted Days

According to Ab.3.6.cd[44]:

शशिदिवसा विज्ञेया बूदिवसोनास्तिथिप्रलयाः॥

lunar days decreased by solar days should
be known as ommited lunar days (*tithis)*||

The number of civil ("solar'') days in the great *yuga* is known, 15 779 175 000. We know from Ab.1.3 and AB.3.6 that there are 57 753 336-4 320 000=53 433 336 lunar months in a *yuga*. The number of lunar days in a great *yuga* is therefore b_{av}=53 433 336x 30= 1 603 000 080.

Therefore the number of omitted lunar days (*avama*) in a great *yuga* is $a_{av=}$1 603 000 080-15 779 175 000=25 082 580. Verse 82 computes a solution of KU(a_{av}, b_{av}, *-1*)[x_{av}, y_{av}], where a_{av} and b_{av} are additionally reduced by 60: y=(418 043x-1)/26 716 668 (6 247 031, 97 749), the smallest positive solution is provided here.

By a reasoning analogous to the one used for planetary pulverizers and intercalary days, one can attempt to interpret the results found, according to the assumption that the number of omitted lunar days during a great *yuga* is proportional to the number of lunar days. Indeed, attempting to make "sense'' of the result, one could interpret it in this way, when 1/b_{av}= 97749+1/126 716 668 omitted days in 6 247 031 lunar days have elapsed, and 97 749 and 1/26 716 668 have additionally been omitted since the beginning of a great *yuga*.

Sun's declination

According to Mbh.6 ab[45]:

इष्टाज्यांमुनिरन्ध्रपुष्करशशिक्षुण्णांसदासंहरेद्
व्यासार्धेनभवेदपक्रमगुणस्...|

Multiply the Rsine of the given longitude
by 1397 and always divide by the radius;
the result is the Rsine of the declination
for that time.

If λ is a longitude at a given time, and δ, the sun's declination then we have 1397/3438=Rsin δ/Rsin λ. We can then understand verse 83 as providing a solution of

KU(a_{apa}, b_{apa}, -1)[x, y],

where a_{apa} is the maximum value of the R sine of the sun's declination in a yuga, 1397, and b_{apa} corresponds to the Radius of the celestial sphere in minutes, 3438.Therefore, when verse 83 provides (347, 141) as the least positive solution of y=(1397x-1)/3438, this would mean that when 1/3438 of the Rsine of the sun's declination is known, we know that the Rsine of the sun's declination at that time is 347+1/3438 and that the longitude is 141.

Intercalary months

Verses 84-91 tabulate solutions of KU(a_{ama}, b, -1)[x, y],

where a_{apa} is the number of intercalary month (*adhimāsa*) in a *yuga*, that is 1 593 336 (according to Ab.1.3 and AB.3.4) and b the number of civil days in a great *yuga*.

Table 15: Subdivisions of Time

Sanskrit	English	Amounts in a Year	Powers of 60
varṣa	Years	1	*1/60 x 60*
māsa	Months	*12*	*1/5 x 60*
divasa	Days	360	*6x60*
n+		21 600	*6x60²*
vin+		1 296 000	*6x60³*
gurvakṣara	A long syllable	77 760 000	*6x60⁴*

A possible astral interpretation of this rule would thus be, knowing that 1/b part of an intercalary month is known to have elapsed, the total number of intercalary months from the beginning of a great *yuga* and the number of days elapsed can be produced.

Verses 84-90 evoke measuring units for the movements of the planets to evoke what theoretically is here the subdivisions of months. Ab.3.2 however specifies the fact that there is an equivalence: having spelled out the subdivisions of time, as shown in Table 25, it notes[46]:

> क्षेत्रविभागस्तथा भगनात्
>
> The division of a circle (lit. the ecliptic) proceeds in a similar manner from the revolution.

Note however that the names given to time do not go as far as the fourths considered here.

Appendix C *Bhūta-saṅkhyā* Glossary

The reference in parentheses indicates the first occurrence of the expression in the verse. "And also'' indicates other synonyms used.

Zero *nabhas* void, sky (verse 3) and also *kha* (verse 3) *viyad*, *viyat* (verse 4), *śunya (verse 7), vyoma*, *gagana*, (verse 9), *abhra* (verse 57).

One *śasin*, "the one who has a rabbit'' the moon (verse 3) and also *śītāṃśu*"cold-rayed" (verse 5) *prāleyaraśmi* "frosty rayed" (verse 17), *himāṃśu* (verse 27), *sudhāmayūkha* (verse 49)

Table 16: Tabulated Values for Intercalary Months

Measuring Units	*d*	a_{apa}	*b in days*	*x*	*y*
Revolutions	12	*132 778*	*131 493 125*	18 900 317	19 085
Signs	12	*1 593 336*	*131 493 125*	122 110 391	1 479 643
Degrees	5	*9 560 016*	*26 298 625*	24 734 836	8 991 551
Minutes	5	*114 720 192*	*5 259 725*	3 814 478	83 197 819
Seconds	5	*1 376 642 304*	*1 051 945*	142 549	186 548 711
Thirds	5	*16 519 707 648*	*210 389*	134 606	10 569 239 683
Fourths	5	*991 182 458 880*	*210 389*	93 412	44 081 638 531

niśānātha “lord of the night/dreams” (verse 58), *rātrināthā* (verse 65); *amṛtasanmayūkkhā* (verse 70).

Two *dasra*, a name of the *aśvin*s (verse 3), and also *yama, nayana* (according to a Shukla inference in verse 3) eyes; *bhuja*, the arms (verse 4), and also *bāhu* (verse 9).

Three Rāma, there are three famous Rāmas: the hero of the Rāmayaṇa, Balarāma (Kṛṣṇa's brother) and Parasurāma (verse 3). *Guṇa* as the three qualities of all created things (truth/goodness for gods (*sattva*), matter/passions for men (*rajas*), darkness/ignorance for demons (*tamas*) (verse 3); *vahni*, or *agni* as the three sacrificial fires (verse 9), and *kṛśānu* (verse 21), *dahana* (verse 62) and *vikrama* as the three steps of Viṣṇu. And also *puṣkara* (verse 9, 83) as the name of three sacred places.

Four *abdhi*, ocean, there are four oceans in puranic cosmology (verse 3), and also *sāgara, ambu* (verse 11), *samudra* (verse 33) *apagānātha* (lord the rivers, i.e ocean) (verse 59), *udadhi* (verse 60); *veda*, like the four (verse 4); *kṛta* as the lucky age in the *yugas*, or the lucky throw in a dice game showing the number four (verse 30).

Five *iṣu*, arrow, as the five arrows of Kāma, the god of love, (verse 3) and also *śara* (verse 3) *bāna* (verse 11); *bhūta*, the five elements (earth, air, fire, water and stone) (verse 4), but also (*tanmātra*) (verse 22); *artha* (verse 37) objects of the five sense organs.

Six *rasa*, perfume, taste. There are six tastes: *kaṭu* (acrid), *amla* (sour, acid), *madhura* (sweet), *lavaṇa* (saline), *tikta* (bitter) and *kaṣāya* (astringent, fragant) (verse 3); *aṅga*, as the six Vedāṅgas.(verse 3); *ṛtu* as six seasons (verse 66).

Seven *naga*, “that which does not move'', a mountain, there are seven chains of mountains according to puraṇic mythology (verse 3) and also *adri* (verse 3), *śaila* (verse 16), *aga* (verse 17), *paravata* (verse 20), *kṣoṇīdharā* (verse 21), *kṣitibhṛt* (verse 41- partly a Shukla addendum) *bhūbhṛt* (verse 62), *śiloccaya; tāpasa* “one who practices austerities”, as a synonym of *muni*, designating the seven stars of Ursa Major (verse 9), *ṛṣi* (verse 17), *aśva* “horse” as the number of horses of the sun (verse 25); *svara* “notes of the musical scale'' (verse 77).

Eight *vasu*, a class of eight deities (verse 3); *gaja* elephant; there are eight elephants symbolizing the eight cardinal directions (East, West, South, North, South-east, South-west, North-east, North-west) (verse 3) and also *kuñjara, mataṅga* (verse 4)*; nāga* (vers 17), *pannaga* (verse 10) *phaṇibhṛt* (verse 71)*; bhujaṅga* for eight mythologcal Nāga kings.

Nine *nanda* either the nine treasures of Kubera or the nine brother-kings called “Nanda'' (verse 3), *aṅka*, the nine digits (verse 4) *randhra*, orifice; the nine orifices of the human body are: the two eyes, the two nostrils, the mouth , the two ears, the sex, the anus. And also *chidra* (verse 26).

Ten *diś* as the ten directions: the four cardinal directions, North, South, East, West, and North west, North-east, South-west, South-east, below, above.

Eleven *śiva*, as the head of a group of eleven gods called collectively *rudra* (verse 3).

Twelve the sun, as the twelve signs of the zodiac, or the twelve months of the solar year?, *Sūrya* (verse 11) *vivasvad* (verse 61).

Thirteen *viśva* as a class of gods containing 1. Vasu , 2. Satya , 3. Kratu , 4. Daksha , 5. Kāla , 6. Kāma , 7. Dhṛiti , 8. Kuru , 9. Purkravas , 10. Mādravas, 11. Rocaka 12. Dhvani 13. Dhūri (according MW) (verse 28).

Fourteen *indra* (verse 85), *śakra* (as a class of gods related to *indra*) (vers 60), *manu* (verse 7).

Fifteen *tithi* as the fifteen *tithis* of the moons increase (and decrease) (verse 45).

Eighteen *dhṛti* a class of metres consisting of 4 x 18 syllables (according to Colebrooke) (verse 38) and also *dhārtu* (verse 14), *saṃskradhṛti* (verse 46).

Twenty-four *jina* 24 Jinas are supposed to flourish in each of the 3 *Avasarpinīs*?? (hence) the number '" 24 "' (verse 79).

Twenty-five *Tattva* (verse 50).

Twenty-seven *ṛkṣa* as a synonym of *nakṣatra* (verse 14).

Thirty-two *danta* (verse 20).

Forty-eight *saṃskāra* (verse 46)- should this account for the number of ceremonies, or the different qualities?

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Notes and References

Datta, B. "Elder Āryabhaṭa's rule for the solution of indeterminate equations of the first degree." *Bulletin of the Calcutta Mathematical Society*, XXIV(1), (1932), 19-36.

Hayashi, T. "*Kuṭṭakaraśiromaṇi of Devaraja, Sanskrit Text with English Translation, Notes and Appendices*" (Reprinted from *IJHS*, 46.1-47-1. (2011-2012) ed.). Indian National Science Academy. (2012).

Hayashi, Takao. "*Bijagaṇita* of Bhāskara". *SCIAMVS*, 10 (2009).

Keller, A. *Expounding the mathematical seed, Bhāskara and the mathematical chapter of the Āryabhaṭīya* (Vol. 2 volumes). Basel: Birkhauser, (2006).

Keller, A. "Qu'est ce que les mathématiques? les réponses taxinomiques de Bhāskara un commentateur, mathématicien et astronome du VIIème siècle". In *M. Hert Phillipe; Paul-Cavalier* (Ed.), *Sciences et frontières* Kimé, (2007) 29-61.

Pingree, D. *Census of the Exact Sciences in Sanskrit* (CESS) (Vol. 5 volumes). American Philosophical Society (1970-1995).

Pingree, D. *Jyotiḥśastra : astral and mathematical literature.* Wiesbaden:Harrassowitz, (1981).

Plofker, K. *Mathematics in India*. Princeton University Press, (2009).

Rao, S. B. *Indian Astronomy, An Introduction*. Univeristies Press (2000).

Sarma, S. R. "On the Rationale of the Maxim *Aṅkānāṃ Vāmato Gatiḥ*". *Gaṇita Bharati*, 31(2009) 65-86.

Sen, S. N. *A Concise History of Science in India*. Dehli, (1971) 58-135.

Sengupta, P. C. *The Aryabhatiyam, translation*, XVI (1927).

Severi, C. *Le principe de la chimère, une anthropologie de la mémoire.* Paris: Aesthetica, Editions de la rue d'Ulm, musée du quai Branly, (2007).

Shukla, K. S. *Mahābhāskarīya, Edited and Translated into English, with Explanatory and Critical Notes, and Comments, etc*. Lucknow, Department of Mathematics Lucknow University, 1960.

Shukla, K. S. (1976). *Āryabhaṭīya of Āryabhaṭa, with the commentary of Bhāskara I and Someśvara*. New-Dehli: Indian National Science Academy, 1976.

Sarma, K.V. and Shukla, K.S. *Āryabhaṭīya of Aryabhaṭa, critically edited with translation*. New-Delhi: Indian National Science Academy, 1976.

Notes

1. This is already noted by Shukla, (Shukla 1976, Introduction, lxxxii).

2. If we accept Shukla's analysis of the text which is sometimes transmitted without the first part considered by some as an independant text. See (Shukla 1976, Introduction, 6.2, xxv-xxvi).

3. He can comment on half a verse, or two verses at times, etc..

4. (Shukla 1976, pp. 5-7).

5. (Keller 2007).

6. Between arithmetics and algebra, (Plofker 2009, p. 296).

7. The «pulverizer» algorithm owes probably its name to the first part of the process in which a repeated division- an Euclidean algorithm- pluverises two numbers into a list of smaller numbers. Bhāskara uses the orthography *kuṭṭākāra*, "what makes a pulverization''. Once in his commentary (Shukla 1976, p. 135, line 1) *kuṭṭāka* is used in a compound-*nir-agra-kuṭṭāka*. Manuscripts should be checked to be sure that this is not an editorial misprint. Later authors indeed seem to have privileged the orthography *kuṭṭāka*, an action noun derived from the verb *kuṭ*-, "to pulverize", whose literal meaning is "a tool to pulverize", hence a "pulverizer".

8. (Shukla 1976, p. 44), (Keller 2007).

9. (Shukla 1976, pp. 136-156) and (Keller 2006, pp. 135-166). Note that topics at the interface between mathematics and astral science may have been used by Āryabhaṭa to make transitions from one chapter to another. Thus the 13th and last verse of the *gītikā* part deals with Sines, another topic which is treated in both domains. However, the procedure derived from the Pythagoran theorem is stated in the middle of the part on mathematics, in verse 17.

10. The 7th century commentator reads these verses as providing two equivalent procedures for two equivalent types of pulverizer problems and methods, a «pulverizer with remainder», and a «puliverizer without remainder» (Shukla 1976) (Keller 2006). Other commentators will do so as well, along very similar lines. See for instance (Hayashi 2012). Much of the mathematical and astral data developed here can be found in this publication..

11. Using the notation developed in Takao Hayashi's edition of the *Bijagaṇita*, (Hayashi 2010, Appendix A, p. 117 sqq).

12. (Datta1932), (Sarma-Shukla 1976), (Keller 2006).

13. That is algebraically $c\alpha=bq+\alpha_0$ and $c\beta=aq+\beta_0$.

14. The solution $(c\alpha, c\beta)$ of KU(a, b, -c)[x,y] is what we will call in the following the smallest positive solution.

15. (Shukla 1976, p. 137) (Keller 2006, p. 135-136). A general rule concerning the use of this short-cut is given in the *Mahābhāskarīya* (I.45-46)

    (Shukla 1960; p. 32-33):

    "रूपंएकमपास्यापि कुट्टाकारः प्रसाध्यते॥
    गुणकारोऽथ लब्धंचराशी स्यातामुपर्यधः॥४५॥
    इष्टेनशेषम हित्यभजेद्दृढाभ्यांशेषम्दिनानिभगणादिचकीर्त्यतेऽत्र॥

    Mbh.I.45-46ab.

    Alternatively, the pulverizer is solved by subtracting one. The upper and lower quantities are the multiplier and quotient. Having multiplied the remainder by each, one should divide the respective products by the abraded <divisor and dividend>. The remainders obtained here are the days etc. and the revolutions etc.''

16. See the specifications at the end of this article, to explain conventions adopted in the translation as well as the abbreviations relating to the pulverizer.

17. Notably in the Līlāvatī (verse 256) and the Bījagaṇita (verse 36 ab), see (Hayashi 2010, p.106).

18. This can be checked in the spreadsheets I am putting online, at http://halshs.archives-ouvertes.fr/halshs-00760116 .

19. Note that in the introductory sentence to the versified table, the expression *graha-kuṭṭakāra* is used ambiguously: it could be the couple of solutions given in each verse as well as the totality of given solutions. But it does not seem to designate either the method, nor just the x's obtained.

20. (Sengupta 1927), (Pingree 1970), (Sen 1971), (Shukla 1976), (Pingree 1981), (Rao 2000), (Plofker 2009), (Hayashi 2012).

21. The numbers in these verses are given with Āryabhaṭa's special device for noting numbers.(Shukla 1976, pp. 6-8), provides edition and translation here:

    युगरविभगणः खयुघु, शशीचयगियिङुशुछ्लृ, कुङिशिबुण्लृख्खृ प्राक्॥
    शनिढुङिवघ्व, गुरुख्रिच्युभ, कुजभद्लिझ्नुखृमभुगुबुधसौराः॥३॥
    चन्द्रोच्चर्जुष्खिध, बुध सुगुशिथृन, भृगुजषबिखुछ्ळ, शेषारकाः॥
    बुफिनचपातविलोमा, बुधाह्न्यजाकोर्दयाच्च लङ्कायाम्॥4॥

    In a yuga, the eastward revolutions of the Sun are 43,20,000; of the Moon, 5,77,53,336; of the Earth, 1, 58,22,37,500; of Saturn, 1,46,564; of Jupiter, 3, 64224; of Mars 22,96,824; of Mercury and Venus the same as those fo the Sun; of the Moon's apogee, 4,88,219; of (the *śighrocca* of) Mercury, 1, 79, 37, 020; of (the *śighrocca* of) Venus, 70, 22,388; of (the *śighrocca* of) the other planets, the sam as those of the Sun; of the Moon's ascending node in the opposite direction (i.e. westward), 2,32,226. There revolutions commenced at the beginning of the sign Aries on Wednesday at sunrise at Laṅka (when it was the commencement of the current yuga.

    These verses stock numerical data, but with one non-numerical increment, an astral body -real or theoretical- and one output, the number of revolutions in a *yuga*.

22. (Sharma-Shukla 1976):

    "रविभुयोगादिवसाः

    The conjunctions of the Sun and the Earth are (civil) days."

    So that the number of civil days in a *yuga* (*b*) is equal to the number of revolutions of the Sun in a *yuga* minus the number of revolutions of the earth in a *yuga:*

    b=1 582 237 500-4 320 000=1 577 917 500.

23. According to Sho Hirose (personal communication), these measuring units are used in other parts of Bhāskara's commentary in a feminin form (*tatparā, pratatparā).* In at least one verse, verse 15, *tatpara* is undoubtly used in a masculin form thought. Most probably however they are to be understood in the feminin, on the model of *kalā* and *liptā*.

24. (Shukla 1976, lxxxii, titles in Appendix II:335-339).

25. These characterizations I owe to Dominique Tournes, Karine Chemla and the larger HTN group.

26. Note that the order of the planets used for the classification of these sub-sets of solutions is not the one of the enumeration of Ab.1.3-4.

27. भागादीनाम्क्रमात्पञ्च [5 प्रवदन्तिमनीषिनः।।

    Wise ones indicate for degrees etc. in due order five [5||48||

28. एते ग्रहकुट्टाकाराः श्लोकैरप्युपनिबध्यन्ते।

    These planet pulverizers (*graha-kuṭṭākara*) are indeed condensed (*upanibādh-*) with verses (*śloka*).

29. (Severi 2007).

30. Studies on this way of expressing numbers are innumerable and were published as early as the XVIIIth century. For the latest insight, see (Sarma 2009).

31. I note here, having not seen this remarked elsewhere, that a standard name for nine, can be seen as a meta-name. Indeed, *aṅka*, "digit" is used here as indicating the total amount of digits: it is the cardinal number of the set of digits that is used to name the number nine. In other words a number of numbers used to designate a number.

32. Of course there need not be a conceptual consistency between the way syntactically numbers and quantities are related and the way they are thought of. But the close study of syntactical expressions, may when they show variations, yield some information in this respect.

33. Note that if etymologically sam-Mī is related to the act of measuring, its current meaning can also be "to fasten things together'', thus a quantity can also be understood here as being expressed by a set of units/ numbers fastened together.

34. The derivation of these values is explained in Example 7 of BAB.2.32-33, (Shukla 1976, pp. 137-138), {(Keller 2006, pp. 135-136).

35. BAB.2.32-33 ex.9 gives the derivation of these values. (Shukla 1976, pp. 138-139), (Keller 2006, p. 138-139).

36. The derivation of these values is explained in BAB.2.32-33.ex.10 (with maybe a misprint of 51 011 rather than 59 011, in the solved equation in both Shukla's edition, and my translation), (Shukla 1976, p. 140), (Keller 2006, pp. 140-141).

37. These values are computed by Bhāskara in BAB.2.32-33 ex11, (Shukla 1976, pp. 141-142), (Keller 2006, pp. 141-143).

38. This last number is the greatest common divisor of the revolution number of the sun, and the number of civil days in a yuga, that is the GCD(4 320 000, 1 577 917 500). Despite the formulation it serves here only as a divisor for the number of revolutions of the sun in a great *yuga*, all other values remain unchanged (or "divided by one'').

39. Shukla corrects सप्ति seven into अष्टि to rectify the manuscripts readings and obtain a coherent numerical result.

40. (Shukla 1976, pp. 335-339).

41. Shukla and Sarma add- (Shukla 1976, pp. 6-7), that it is is actually a mean value in which the mean revolutions of Mercury around the earth are equated to the mean revolutions of Mercury around the sun (*śīghrocca*). But this is not at stake in the present data considered.

42. The verse actually gives a more general rule, this is but one of its interpretations, see (Shukla 1976, p. 87).

43. (Shukla 1960, skt 4, eng. 17)

44. (Shukla 1976, p.91).

45. (Shukla, 1960, skt 11, eng 62)

46. (Shukla 1976, p. 85).