

## SOME APPLICATIONS OF FIRST APPROACHABLE ŚUNYA AND DERIVATION OF OTHER APPROACHABLE ŚUNYAS

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### Abstract

A real number, expressed as decimal string, can be considered as the sum of infinite terms of rational numbers. Since the cardinality of rational numbers is only  $\aleph_0$ , these strings cannot go beyond  $\aleph_0$  places of decimals. This puts limitation on the result of multiplication of two real numbers  $\aleph_0$  decimals long. The squaring of the decimal string 0.333... and comparing it to the value of 1/9 shows that the concept of First Approachable Śunya (FAS) must be called upon to explain the limiting value obtained as answer to this and similar problems. Further, a combinatorial argument shows that the cardinality of real numbers is  $10^{\aleph_0} = 2^{\aleph_0}$  and the first member of the set of real numbers must be the FAS ( $1/10^{\aleph_0}$ ). Two theorems for raising any real number to  $\aleph_0$  are derived and their implication in bolstering of Continuum Hypothesis is highlighted. Eight-by-ten and five-by-ten Cantor sets are used to understand the real numbers from 0.000...0 to 0.999...9. Also, more Approachable Śunyas could be conceived based on division of 1 into smallest possible value. The second order Approachable Śunyas could be derived based on exponentiation (square roots).

**Key words:** Approachable Śunya, Cardinality, Division, Exponentiation, Real Numbers

### 1. Introduction

A decimal string 0.123 can be considered

as the sum of  $\frac{1}{10} + \frac{2}{10^2} + \frac{3}{10^3}$ , all of them rational

numbers. What is the maximum number of digits up to which a decimal string can extend? This is the same as asking the question: what is the maximum length of decimal places up to which any real number between 0 and 1 can go? Consider the rational representation of real numbers as

$\frac{a_x}{10^1} + \frac{a_x}{10^2} + \frac{a_x}{10^3} + \dots$ , where  $a_x$  is any digit from 0

to 9. When the number of these terms reaches  $\aleph_0$ , the denominator reaches  $10^{\aleph_0}$ . The denominator of the next term must be  $10^{(\aleph_0+1)}$ . But  $\aleph_0 + 1 =$

$\aleph_0$  and hence we are forced to conclude that the maximum number of digits of any real number between 0 and 1 can only be  $\aleph_0$  and nothing higher.

When two decimal numbers (e.g. 0.2 and 0.3) are multiplied, their answer adds up the number of decimal places of the multiplicand and multiplier. Thus  $0.2 \times 0.3 = 0.06$ , and  $0.02 \times 0.03 = 0.0006$ . Now what happens when two decimal strings  $\aleph_0$  digits long are multiplied? Obviously the number of digits of the answer should be  $\aleph_0 + \aleph_0$ . But  $\aleph_0 + \aleph_0 = \aleph_0$  and so the answer can only be  $\aleph_0$  digits long. Does it mean that the answer is truncated to the first  $\omega$  decimal places – where  $\omega$

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**Table 1:** Pattern of  $(0.333\dots)^2$

Decimal string (x)	No. of decimal places of (x)	(x) <sup>2</sup>	No. of decimal places of (x) <sup>2</sup>
0.3	1	0.09	2
0.33	2	0.1089	4
0.333	3	0.110889	6
0.3333	4	0.11108889	8
0.33333	5	0.1111088889	10
0.333...	$\aleph_0$	$0.\underbrace{111\dots0888\dots}_9$	$\aleph_0 + \aleph_0$

is the ordinal of cardinal  $\aleph_0$ ? Or does it mean that the answer is rounded to the first  $\omega$  decimal places? Does it put any limitation on the accuracy of the answer? These issues are examined using the example of 0.333...

**2. Squaring of 0.333...**

In an earlier paper (Basant & Panda, 2013), it was shown that the actual sum of all convergent geometric series of rational numbers is less than the ideal sum by  $\frac{1}{2^{\aleph_0}}$ , defined as FAS. As shown in Section 2 of that paper,  $2^{\aleph_0} = 10^{\aleph_0} = 3 \times 10^{\aleph_0}$  etc. and so  $\frac{1}{2^{\aleph_0}} = \frac{1}{10^{\aleph_0}} = \frac{1}{3 \times 10^{\aleph_0}}$  and so on. It

was therefore shown that  $\frac{1}{3} = 0.333\dots + \frac{1}{2^{\aleph_0}}$  to be more accurate. In the above paper, the FAS was arrived at based on the remainder of Left Hand Side (LHS) – Right Hand Side (RHS) after summation of  $\aleph_0$  terms. It is possible to test the validity of the theorem of FAS in the squaring of 0.333....

Let us start by first assuming that,  $\frac{1}{3} = 0.333\dots$

Squaring both sides, we get

$$\frac{1}{9} = (0.333\dots)^2 \quad \dots(1)$$

Now, a number, whose digits consist of 3s alone, when squared, can only result in a new number that ends in 9 (whatever that number may be). Or we can write the RHS as 0.A9 where A9 is a decimal string of  $\aleph_0 + \aleph_0 = \aleph_0$  digits. Now let

us find the value of LHS.  $\frac{1}{9} = 0.111\dots$ . Here is a number whose only digits are 1s and so we can write it as 0.B1, where 0.B1 is a decimal string of  $\aleph_0$  number of 1s. Thus, we can rewrite equation (1) as 0.B1 = 0.A9. But  $1 \neq 9$  and so this identity obviously is not correct. The implication is that our assumption  $\frac{1}{3} = 0.333\dots$  as well as  $\frac{1}{9} = 0.111\dots$  are not accurate enough.

Instead of writing the RHS as 0.A9 it is possible to find out its exact value. This is given in Table 1.

Now we can rewrite equation (1) as

$$0.\underbrace{111\dots}_{\aleph_0} = 0.\underbrace{111\dots0888\dots}_{\aleph_0}9 \quad \dots(2)$$

Here the LHS is  $\aleph_0$  digits long and the RHS is  $\aleph_0 + \aleph_0$  digits long. Even if the RHS is truncated to  $\omega$  digits, LHS = 0.111...1 and RHS = 0.111...0. Thus LHS  $\neq$  RHS. And so we have a problem.

**Table 2.** Pattern of  $(0.666\dots)^2$

Decimal string (x)	No. of decimal places of (x)	$(x)^2$	No. of decimal places of $(x)^2$
0.6	1	0.36	2
0.66	2	0.4356	4
0.666	3	0.443556	6
0.6666	4	0.44435556	8
0.66666	5	0.4444355556	10
0.666...	$\aleph_0$	$0.444\dots3555\dots6$	$\aleph_0 + \aleph_0$

It is not difficult to point out similar examples with easily understandable patterns.

**Example 1**

Take  $\frac{2}{3} = 0.666\dots$ . Squaring both sides,  $\left(\frac{2}{3}\right)^2 = (0.666\dots)^2$ . The possible final digit of RHS is 6.

However LHS =  $\frac{4}{9} = 0.444\dots4$ , where the last possible digit can only be 4. But  $4 \neq 6$  and so  $LHS \neq RHS$ . Or, Table 2 can be used to derive the exact value of  $(0.666\dots)^2$ :

The relevant equation therefore becomes:

$$0.\underbrace{444\dots}_{\aleph_0} = 0.\underbrace{444\dots3555\dots6}_{\aleph_0 \quad \aleph_0} \dots(3)$$

LHS and RHS obviously do not match. Let us look at another example.

**Example 2**

$\frac{2}{3} = 0.666\dots$  and  $\frac{1}{3} = 0.333\dots$ . Therefore  $\frac{2}{3} \times \frac{1}{3} = 0.666\dots \times 0.333\dots$  or  $\frac{2}{9} = 0.X8$ , where X8 is some

decimal string. But  $\frac{2}{9} = 0.222\dots2$ . Here  $8 \neq 2$  and so  $LHS \neq RHS$ . Or as in the previous example, the exact value of  $0.666\dots \times 0.333\dots$  can be obtained from Table 3.

The relevant equation in this case is equivalent to:

$$0.\underbrace{222\dots}_{\aleph_0} = 0.\underbrace{222\dots1777\dots8}_{\aleph_0 \quad \aleph_0} \dots(4)$$

**Table 3:** Pattern of  $0.666\dots \times 0.333\dots$

Decimal string (x)	Decimal string (y)	Sum of decimal places of (x) + (y)	$(xy)$	No. of decimal places of $(xy)$
0.6	0.3	1+1	0.18	2
0.66	0.33	2+2	0.2178	4
0.666	0.333	3+3	0.221778	6
0.6666	0.3333	4+4	0.22217778	8
0.66666	0.33333	5+5	0.2222177778	10
0.666...	0.333...	$\aleph_0 + \aleph_0$	$0.222\dots1777\dots8$	$\aleph_0 + \aleph_0$

**Table 4:** Pattern of  $0.0909\dots \times 0.111\dots$

Decimal string (x)	Decimal string (y)	Sum of decimal places of (x) + (y)	(xy)	No. of decimal places of (xy)
0.09	0.11	2+2	0.0099	4
0.0909	0.1111	4+4	0.01009899	8
0.090909	0.111111	6+6	0.010100989899	12
0.09090909	0.11111111	8+8	0.0101010098989899	16
0.0909...	0.111...	$N_0 + N_0$	$0.0101\dots009898\dots99$	$N_0 + N_0$

Here also  $LHS \neq RHS$ . We can think of yet another example.

**Example 3**

Take  $\frac{1}{11} = 0.0909\dots$  and  $\frac{1}{9} = 0.111\dots$

Multiplying,  $\frac{1}{11} \times \frac{1}{9} = 0.0909\dots \times 0.111\dots$  or

$\frac{1}{99} = 0.Y9$ , where Y9 is some decimal string. But

$\frac{1}{99} = 0.0101\dots$ . As  $1 \neq 9$ ,  $LHS \neq RHS$ . Here too, the exact value of  $0.0909\dots \times 0.111\dots$  can be derived from the pattern of Table. 4:

Thus the relevant equation turns out to be:

$$\underbrace{0.010101\dots}_{N_0} = \underbrace{0.0101\dots009898\dots99}_{N_0} \dots(5)$$

Obviously, LHS and RHS of equations (2) to (5) do not match. What could be an explanation for this? Could FAS throw some light on the situation?

**3. Application of First Approachable Sunya**

Let us discuss equation (2). Using Theorem of FAS (Basant & Panda, 2013) let us

denote  $\frac{1}{3} = 0.333\dots + \frac{1}{2^{N_0}}$ . Now  $\frac{1}{2^{N_0}}$  is the

*equivalent* reminder and the *actual reminder* is

$$\frac{1}{3 \times 10^{N_0}}. \text{ Thus } \frac{1}{3} = 0.333\dots + \frac{1}{3 \times 10^{N_0}}.$$

Squaring both sides we get

$$\frac{1}{9} = 0.111\dots0888\dots9 + \frac{2 \times 0.333\dots}{3 \times 10^{N_0}} + \left(\frac{1}{3 \times 10^{N_0}}\right)^2.$$

But  $\frac{1}{9} = 0.111\dots + \frac{1}{9 \times 10^{N_0}}$ . Thus we have,

$$0.111\dots + \frac{1}{9 \times 10^{N_0}} = 0.111\dots0888\dots9 +$$

$$\frac{2 \times 0.333\dots}{3 \times 10^{N_0}} + \frac{1}{9 \times (10^{N_0})^2}$$

$$\text{Or, } 0.111\dots + \frac{1}{9 \times 10^{N_0}} = 0.111\dots0888\dots9 +$$

$$\frac{0.222\dots}{10^{N_0}} + \frac{1}{9 \times (10^{N_0})^2}$$

Expanding  $\frac{1}{9}$  on the LHS again, we can write

$$0.111\dots + \frac{0.111\dots + \frac{1}{9 \times 10^{N_0}}}{10^{N_0}} =$$

$$0.111\dots0888\dots9 + \frac{0.222\dots}{10^{N_0}} + \frac{1}{9 \times (10^{N_0})^2},$$

$$\text{Or } 0.111\dots + \frac{0.111\dots}{10^{\aleph_0}} + \frac{1}{9 \times (10^{\aleph_0})^2} =$$

$$0.111\dots 0888\dots 9 + \frac{0.222\dots}{10^{\aleph_0}} + \frac{1}{9 \times (10^{\aleph_0})^2}$$

Removing the common term from both sides,  
 $0.111\dots + \frac{0.111\dots}{10^{\aleph_0}} = 0.111\dots 0888\dots 9 + \frac{0.222\dots}{10^{\aleph_0}}$ .

Here we must understand the formalism of the expression  $\frac{0.111\dots}{10^{\aleph_0}}$ . What does  $\frac{5}{10^4}$  mean? It means 0.0005. Or, in other words when a number is divided by  $10^x$ , it can be expressed as a decimal string with x digits. Look at Table 5.

**Table. 5:** Pattern of division by powers of 10

Rational representation	Decimal representation	Number of decimal places
$\frac{1}{10^1}$	0.1	1
$\frac{1}{10^2}$	0.01	2
$\frac{1}{10^3}$	0.001	3
$\frac{1}{10^{\aleph_0}}$	0.000...1	$\aleph_0$
$\frac{0.1}{10^{\aleph_0}}$	0.000...01	$\aleph_0 + 1$
$\frac{0.111\dots}{10^{\aleph_0}}$	0.00...0111...	$\aleph_0 + \aleph_0$

So if y is any integer then  $\frac{y}{10^{\aleph_0}}$  represents a decimal string that is  $\aleph_0$  digits long. It can also be expressed as 0.000...y (say A) which is  $\aleph_0$  decimal places long. Similarly, if 0.z is any decimal string, then  $\frac{0.z}{10^{\aleph_0}}$  is equivalent to

0.000...(Say B), where there are  $\aleph_0$  zeros following the decimal point but preceding z. Now A is the decimal-part and B is the Śunya-part. Both are  $\aleph_0$  digits long. Using this formalism, LHS =  $0.111\dots + \frac{0.111\dots}{10^{\aleph_0}} = 0.111\dots:111\dots$  up to  $\aleph_0 + \aleph_0$  places of decimals. Here the colon (:) separates the Śunya-part from the decimal-part. Further, RHS =  $0.111\dots 0:888\dots 9 + \frac{0.222\dots}{10^{\aleph_0}}$  is equivalent to  $0.111\dots 0 + \frac{0.888\dots 9 + 0.222\dots 2}{10^{\aleph_0}} = 0.111\dots:111\dots$  up to  $\aleph_0 + \aleph_0$  places of decimals.

Thus, when  $\aleph_0 + \aleph_0$  places of decimals are considered, we have LHS = RHS. Once this identity is established, we can discard the Śunya-part of the final answer (since it is zero in one way of reckoning and since  $\aleph_0 + \aleph_0 = \aleph_0$ ) and retain the decimal-part. Now the answer will be correct up to  $\aleph_0$  decimals.

It can similarly be shown that equations (3) to (5) mentioned in Examples (1) to (3) can also be balanced using FAS. Here it is not the constant form of the FAS, i.e.,  $\frac{1}{2^{\aleph_0}}$  that is used but the actual remainder of  $IS \times r^{\aleph_0}$  (where IS = Ideal Sum and r is the common ratio between adjacent terms of the geometric series) as shown in Section 3 of the earlier paper (Basant & Panda, 2013). Thus in the case of equations (3) to (5) also, when  $\aleph_0 + \aleph_0$  decimal places are considered, LHS = RHS. And as in the case of equation (2), we can then discard the Śunya part and retain the real part of  $\aleph_0$  decimals.

**Recounting 1**

...Into the court of Emperor Asoka came three brothers with a strange request. Their father had bequeathed them 17 cows to be divided among

the three in the ratio  $\frac{1}{2} : \frac{1}{3} : \frac{1}{9}$ . They could not do it on their own and so had come to the Royal Court. The Emperor looked at the court mathematician who rose to his feet immediately and pronounced

the solution:  $8\frac{1}{2}$  cows for the eldest brother,  $5\frac{2}{3}$

cows for the next brother and  $1\frac{8}{9}$  cows for the

youngest one. “ $8\frac{1}{2}$  and  $5\frac{2}{3}$  and  $1\frac{8}{9}$  cows?”,

asked the perplexed Emperor. “Your Majesty, the calculation gives that solution,” said the mathematician. “So you want to butcher three cows to satisfy your calculation?” commented the annoyed Emperor. “I say your solution is truly improper. It is even worse than the problem”, he did not hide his displeasure. He was upset and retired for the day after directing the brothers to come the next day.

That night the Emperor prayed to Lord Buddha to enlighten his understanding. And in his dream he heard the Divine Voice: “Gift the Royal Cow”. The Emperor was puzzled, but decided to obey the command implicitly. So the next day he gifted the Royal Cow to the startled brothers.

They returned to their homes full of trepidations. They were convinced that the Emperor had punished them. For if they did not maintain the Royal Cow in royal style, then the Emperor would be displeased and if they maintain it the way it deserves, then they will go pauper in no time. It was with a sad heart that they entered their homes.

Seeing their despondency, the intelligent and modest wife of the eldest brother hinted to her husband tangentially: “The Emperor has given you the Royal Cow with the command to **solve your problem**. Not attempting to do so would be disobeying the royal command.” So the brothers reluctantly set about dividing the cows.

Now there were  $17+1=18$  cows and the eldest brother got 9 cows, the next one got 6 cows and the youngest one got 2 cows.  $9+6+2=17$  and so one cow was left over. Each of them was intelligent enough not to claim the Royal Cow as his share.

The same day the three brothers hurried to the Royal Court to thank the Emperor and return the Royal Cow...

Ebullient Lilliputians enter the garden to frolic  
But make themselves scarce after committing the havoc;  
Distraught gardener would look at the decimated entities  
And frown at the ‘ghosts of departed quantities’

(Berkeley, 1734)...

#### 4. Some Explanations

Some conclusions can be drawn from the exercise :

- 1) The statement  $\frac{1}{3} = 0.333\dots$  is only approximately correct and this validates the theorem of FAS that “The actual sum of any convergent geometric series of rational numbers (where  $r < 1$ ;  $r = \frac{1}{n}$  where  $n \in N$ ;  $n > 1$ ) is less than its ideal sum by  $\frac{1}{2^{N_0}}$ , which is the First Approachable *Śunya*” (Basant & Panda, 2013).
- 2) When two decimal strings (that are convergent geometric series) of  $N_0$  digits long are multiplied, the answer is not the truncated value up to  $\omega$  digits (As shown using equation: 2 - 5). Further, it can be noticed that when the initial  $\omega$  digits are considered, the difference of LHS – RHS is equal to  $\frac{1}{10^{N_0}}$  as shown in Table 6. This difference of  $\frac{1}{10^{N_0}}$  is, of course, equivalent to  $\frac{1}{2^{N_0}}$ , the FAS.



**Table 6:** Pattern of digits up to  $\omega$  decimal places

Equation No:	LHS (up to $\aleph_0$ ( $\omega$ ) decimals)	RHS (up to $\aleph_0$ ( $\omega$ ) decimals)	LHS – RHS (Decimal)	LHS – RHS (Rational)
(2)	0.111...1	0.111...0	0.000...1	$\frac{1}{10^{\aleph_0}}$
(3)	0.444...4	0.444...3	0.000...1	$\frac{1}{10^{\aleph_0}}$
(4)	0.222...2	0.222...1	0.000...1	$\frac{1}{10^{\aleph_0}}$
(5)	0.0101...01	0.0101...00	0.000...1	$\frac{1}{10^{\aleph_0}}$

**Table 7:** Pattern of  $(0.999\dots)^2$

Decimal string (x)	No. of decimal places of (x)	(x) <sup>2</sup> places of (x) <sup>2</sup>	No. of decimal
0.9	1	0.81	2
0.99	2	0.9801	4
0.999	3	0.998001	6
0.9999	4	0.99980001	8
0.99999	5	0.9999800001	10
0.999...	$\aleph_0$	$0.999\dots8000\dots1$	$\aleph_0 + \aleph_0$

3) Although the portion beyond  $\omega$  decimal places is, truly speaking, the Śunya part and is only a fraction of the FAS (or Zero), it cannot be discarded at the beginning of the calculation. It does contribute to the final answer by adding  $\frac{1}{10^{\aleph_0}}$  to the digit at position  $\omega$ . On completion of the multiplication, the Śunya part can be neglected.

4) It can be seen that acceptance of the identity  $1 = 0.999\dots$  would lead to the result that  $1^2 = (0.999\dots)^2$ ,  $1^3 = (0.999\dots)^3$  etc. From the table below,  $(0.999\dots)^2$  can be shown to be equal to  $0.999\dots8:000\dots1$

The same result can be arrived at using the theorem of FAS:

$$(0.999\dots)^2 = \left(1 - \frac{1}{10^{\aleph_0}}\right)^2 = 1 - 2 \times \frac{1}{10^{\aleph_0}} + \frac{1}{(10^{\aleph_0})^2} = 0.999\dots8:000\dots1 \quad \dots(6)$$

Similarly, from Table. 8, it can be seen that

$$(0.999\dots)^3 = 0.999\dots7:000\dots2:999\dots9 \quad \dots(7)$$

This also can be seen using the theorem of FAS as given below:

$$\begin{aligned} (0.999\dots)^3 &= \left(1 - \frac{1}{10^{\aleph_0}}\right)^3 \\ &= 1 - 3 \times 1^2 \times \frac{1}{10^{\aleph_0}} + 3 \times 1 \times \frac{1}{(10^{\aleph_0})^2} - \frac{1}{(10^{\aleph_0})^3} \\ &= \left(1 - \frac{3}{10^{\aleph_0}}\right) + \frac{0.000\dots3}{10^{\aleph_0}} - \frac{1}{(10^{\aleph_0})^3} = 0.999\dots7 \end{aligned}$$

**Table 8:** Pattern of  $(0.999\dots)^3$

Decimal string (x)	No. of decimal places of (x)	$(x)^3$	No. of decimal places of $(x)^3$
0.9	1	0.729	3
0.99	2	0.970299	6
0.999	3	0.997002999	9
0.9999	4	0.999700029999	12
0.999...		$0.\underbrace{999\dots7000\dots2999\dots}_9$	$\aleph_0 + \aleph_0 + \aleph_0$

$$+ \frac{0.000\dots3}{10^{\aleph_0}} - \frac{1}{(10^{\aleph_0})^3}$$

$$= 0.999\dots7:000\dots2:999\dots9$$

Thus, if we accept that  $1 = 0.999\dots$ , then we will be hard-pressed to explain the curious identity  $1 = 0.999\dots8:000\dots1 = 0.999\dots7:000\dots2:999\dots9$  etc.

5) In Section 6 of the earlier paper (Basant & Panda, 2013) it was argued that if  $x = 0.999\dots9$ , then  $10x = 9.999\dots0$  and not  $9.999\dots9$ . It was mentioned as a case of the end digit being restrained (as 0) but the middle or left portion expanding (as string of 9s). A similar pattern can be seen in the RHS of equations (2) – (5), while considering the initial  $\aleph_0(\omega)$  digits and the next  $\aleph_0$  (i.e.,  $\omega+\omega$ ) digits.

Such pattern can be seen in the final rows of Table 7 and 8 as well as in the RHS of equations (6) and (7) also.

### 5. Cardinality of real numbers as a permutation

As mentioned in Section 1, any real number between 0 and 1, represented as a decimal string can be considered as the sum of  $\aleph_0$  rationals

$$\left( \sum_{n=1}^{n=\aleph_0} \frac{a_x}{10^n} \right), \text{ where } a_x \text{ is any digit from 0 to 9) and}$$

can thus be only  $\aleph_0$  decimals long. Let us first compute the cardinality of real numbers in the interval between 0 and 1 using a combinatorial argument that is more intuitive than the power set one.

Each of the  $\aleph_0$  decimal places of a real number can be occupied by any of the 10 digits from 0 to 9. Hence the total number of permutations of the decimal places is  $10^{\aleph_0}$ . But  $10^{\aleph_0} = 2^{\aleph_0}$  and thus cardinality of real numbers or the cardinality of all possible decimal strings  $r$  such that  $0 < r < 1$  is  $2^{\aleph_0}$ . (A similar argument

**Table 9:** Pattern of restrained end-digits and expanding digit strings

Equation	End digit of first $\aleph_0(\omega)$ digits	Expanding portion of First $\aleph_0(\omega)$ digits	End digit of second $\aleph_0(\omega)$ digits	Expanding portion of second $\aleph_0(\omega)$ digits
2	0	111...	9	888...
3	3	444...	6	555...
4	1	222...	8	777...
5	00	0101...	99	9898...



**Table 10:** List of Real Numbers

Ordinal number	Corresponding real number	Rational representation
1 <sup>st</sup>	0.000.....0	$\frac{0}{10^{\aleph_0}}$
2 <sup>nd</sup>	0.000.....1	$\frac{1}{10^{\aleph_0}}$
3 <sup>rd</sup>	0.000.....2	$\frac{2}{10^{\aleph_0}}$
4 <sup>th</sup>	0.000.....3	$\frac{3}{10^{\aleph_0}}$
5 <sup>th</sup>	0.000.....4	$\frac{4}{10^{\aleph_0}}$
6 <sup>th</sup>	0.000.....5	$\frac{5}{10^{\aleph_0}}$
7 <sup>th</sup>	0.000.....6	$\frac{6}{10^{\aleph_0}}$
8 <sup>th</sup>	0.000.....7	$\frac{7}{10^{\aleph_0}}$
9 <sup>th</sup>	0.000.....8	$\frac{8}{10^{\aleph_0}}$
10 <sup>th</sup> ( $10^1$ )	0.000.....9	$\frac{9}{10^{\aleph_0}}$
100 <sup>th</sup> ( $10^2$ )	0.000....99	$\frac{99}{10^{\aleph_0}}$
1000 <sup>th</sup> ( $10^3$ )	0.000..999	$\frac{999}{10^{\aleph_0}}$
$10^{\aleph_0}$ -th	0.999.....9	$\frac{999...}{10^{\aleph_0}}$

was used in Section: 8 of earlier paper (Basant & Panda, 2013) to show that the number of possible arrangements of a combination lock with  $\aleph_0$  dials, each with 10 digits from 0 to 9, is  $10^{\aleph_0} = 2^{\aleph_0}$ . Armed with this insight we can make a list of these  $2^{\aleph_0}$  real numbers ( $r : 0 < r < 1$ ) as given in Table:10.

Generation of numbers in Column: 2 of Table: 10 can be explained this way. The interval between 1 and 0 is divided into 10 equal parts. Now each of these parts is  $\frac{1}{10} = 0.1$  and  $10^1$  such parts can be represented from 0.9 to 0.0. Divide

each of them again by 10. Now each new part is  $\frac{0.1}{10} = 0.01$  and each of the  $10^2$  parts can be represented from 0.99 to 0.00. (If a decimal string of two places is to be filled up by all permutations of integers 0 to 9, then  $10^2$  such permutations are possible from 0.00 to 0.99). Now dividing 0.01 again and again by 10, we can make the interval smaller and smaller. When this process is repeated  $\aleph_0$  times, we will reach  $\frac{1}{10^{\aleph_0}} = 0.000...1$  as the smallest possible part. Also, there will be  $10^{\aleph_0} = 2^{\aleph_0}$  such parts represented from 0.000...0 to 0.999...9. (it is easy to see that permutations of integers 0 to 9 for  $\aleph_0$  digits would include all the combinations from 000...0 to 999...9). What happens if we repeat the process one more time? We have  $\aleph_0 + 1 = \aleph_0$ , and so  $\frac{1}{10^{\aleph_0+1}} = \frac{1}{10^{\aleph_0}}$ .

Therefore  $\frac{1}{10^{\aleph_0}} = 0.000...1$  cannot be further divided to get smaller values and so the number of possible decimal strings between 0 and 1 is only  $10^{\aleph_0} = 2^{\aleph_0}$ .

From Table 10 it is evident that we are making a list of the  $2^{\aleph_0}$  real numbers. Now compare this with Cantor’s (Byers, 2007) dazzling diagonal argument which says that such a list is impossible to make. However there is no contradiction since any list can contain items only up to  $\aleph_0$  and not up to  $2^{\aleph_0}$ , *if we proceed through that list one step at a time*, as Cantor meant. Still, if we proceed through the list in blocks of  $10^x$  at a time such that  $x = 1,2,3...etc.$  up to  $\aleph_0$ , then it is evident that after  $\aleph_0$  steps we would reach  $10^{\aleph_0} = 2^{\aleph_0}$ .

A different way to express the list of Table: 10 is to say that the real numbers between 0 and 1 consist of the numbers  $\frac{0}{10^{\aleph_0}}, \frac{1}{10^{\aleph_0}}, \frac{2}{10^{\aleph_0}}, \frac{3}{10^{\aleph_0}}$  etc., up to  $\frac{999...9}{10^{\aleph_0}}$ , where 999...9 is an integer

$\aleph_0$  digits long. Or we can say that the set of real numbers between 0 and 1 consist of the natural numbers (including 0) divided  $\aleph_0$  times by 10, (i.e., by  $10^{\aleph_0}$ ). It shows that the set of real numbers is a well ordered one with the least element (apart from  $\frac{0}{10^{\aleph_0}} = 0.000\dots 0$ ) as  $\frac{1}{10^{\aleph_0}} = 0.000\dots 1$ , the FAS. This incidentally justifies the intuition of Cantor (Foreman & Kanamori, 2010) that “It is always possible to bring any *well-defined* set into the form of a *well-ordered* set” or that all non-empty sets must have a least element.

How many real numbers are there between the points 3.0001 and 3.0002 on the numberline? These two numbers have 4 decimal places. Add  $\aleph_0$  trailing zeroes to them so that they are now 3.0001000...0 and 3.0002000...0 respectively. Now  $4 + \aleph_0 = \aleph_0$ , and so the trailing  $\aleph_0$  digits can be arranged in  $10^{\aleph_0} = 2^{\aleph_0}$  permutations. Thus there are  $2^{\aleph_0}$  real numbers between 3.0001 and 3.0002 and these range from 3.0001000...0 to 3.0001999...9. Similarly it can easily be seen that between the points 0.0 and 0.1 on the numberline there are  $2^{\aleph_0}$  real numbers.

This last observation affords us a way to understand the diagonal argument of Cantor. This observation is equivalent to saying that if we go through the numbers listed in Table: 10 starting from 0.000...0, then we will be covering  $2^{\aleph_0}$  steps before reaching 0.1. We would have by now listed  $2^{\aleph_0}$  real numbers ( $r$ ) but it is obvious that numbers like 0.111...1, 0.222...2, 0.333...3, and of course 0.999...9, which are such that  $0.1 < r < 1.0$ ) will not be in that list.

For the  $2^{\aleph_0}$  real numbers between 0.0 and 0.1, the first digit after decimal point will be 0. This will be followed by all permutations of digits from 000...0 to 999...9. This can be expressed as

0.0x where x is any of the above permutations from 000...0 to 999...9.. Now 0.0x can be put into one-to-one correspondence with 0.1x, 0.2x etc. up to 0.9x. But 0.1x represents all the  $2^{\aleph_0}$  real numbers between 0.1 and 0.2 and 0.2x represents all the  $2^{\aleph_0}$  real numbers between 0.2 and 0.3 etc. It is therefore evident that all the 10 intervals of 0.1 each between 0.0 and 1.0 contain an equal number of real numbers,  $2^{\aleph_0}$ . Thus the cardinality of real numbers between 0.0 and 1.0 is equal to  $10 \times 2^{\aleph_0} = 2^{\aleph_0}$ , the cardinality of real numbers between 0.0 and 0.1.

## Recounting 2

...As shipwrecked Pi Patel (Martel, 2001) drifted in a tiny lifeboat in the Pacific Ocean, he was confronted by two sets of problems.

On the one hand, he had to grapple with the rocking infinite water and reach the shore safely. But the infinity of the sky seemed to him to be a greater danger... As the ethereal witches stirred the cauldron of the sea with bone-white brooms of lightning, as their derisive laughter rose to a thunderous crescendo, as the sea seemed to froth and boil over, as the water seemed to merge with the sky in a phantasmagoria of doom and as the hissing waves threatened to shatter the lifeboat and everything therein, it was indeed herculean struggle to cling on to one's life or sanity.

Similarly, at high noon, when the Sun twitched his whiskers and glared at the sea below with an electric countenance, then every visible thing – including tiny wavelets – turned into mirrors reflecting heat and light. In such blinding heat and light it was indeed difficult to discern the boundary of things. Thus during both these times of cardinality-destroying (multiplicity-erasing) sea-sky continuum, Pi could hardly distinguish his own body from the boat or water – or even worse – Richard Parker.

Apart from this continuum problem, the second set of troubles that confronted Pi revolved around Richard Parker, the tiger on board. Its hunger had to be appeased at regular intervals and yet it was growling, “charlatan”, “renegade”, “corrupter of youth” (Dauben, 1990) etc. and was always threatening to go for the jugular. Keeping an alert and safe distance from the tiger and yet outsmarting it at every instant was the second set of challenges that Pi faced 24 hours a day.

In the midst of these tremendous tasks, if Pi hadn’t noticed few interesting algae drifting under his feet, then O, gentle folks, kindly forgive him, for he was grappling with existential issues of the extremest kind...

Armed with axe and inner compass  
 Pathbreakers wade through bramble and doubt;  
 Roadmakers descend with determined steps  
 And convert their paths to rigid pavements...

**6. Place-value notation and countable infinity**

An argument similar to the one used in Section: 5 to determine the number of digits of a real number can show that the ubiquitous place-value notation can represent natural numbers only up to  $\aleph_0$  digits. With  $\aleph_0$  digits, the value of the digit on the extreme left is  $10^{\aleph_0}$ . Now let us multiply such a number by 10. Therefore the value of the left-most digit must be  $10^{\aleph_0} \times 10 = 10^{\aleph_0+1}$ . But  $10^{\aleph_0+1} = 10^{\aleph_0}$ , and so the value of the new number will be the same as the value of old one. Thus adding digits to a number with  $\aleph_0$  digits does not increase its value and the place-value notation has validity only up to  $\aleph_0$  digits.

The blackhole-cardinal imposes itself on neighbours  
 Nothing can get away from its Dhritarashtra-embrace;

Addition and multiplication cannot cross the event horizon  
 Only exponents manage to power-vault with escape velocity...

It further shows that if we wish to count up to  $\aleph_0$ , – an infinity that is tinier than  $10^{\aleph_0}$  – then the efficient place-value-notation-based numbers (like one hundred, one hundred one... one thousand, one thousand one etc.) are not going to take us there. Such numbers will inevitably take us to  $10^{\aleph_0} = 2^{\aleph_0}$  only. Therefore, to count up to  $\aleph_0$ , we will need a *less efficient* number name and notation.

Could changing the base of the place value system overcome this difficulty? We know that  $10^{\aleph_0} = 12^{\aleph_0} = 60^{\aleph_0} = (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0}$ . Therefore, changing the base to 2 or 12 or 60 or any finite number and counting up to infinity will not take us to  $\aleph_0$ ; it will simply take us to  $2^{\aleph_0}$  only. Hence to count from 0 to  $\aleph_0$ , the counting numbers must be *devoid of any base*. Perhaps a system where 100 is represented by one hundred vertical strokes, and 1000 by one thousand vertical strokes etc. all the way up to  $\aleph_0$  could be imagined. And the names of these counting numbers should also be distinct all the way from 0 to  $\aleph_0$ . In such a clumsy and ‘baseless’ notation alone can we legitimately say that the ‘countable infinity’ is  $\aleph_0$ .

From this discussion it is evident that the concept of ‘countable infinity’ depends on *how* we count.

**7. Different countable infinities**

How can we count up to higher cardinals? Let us look at the numberlines (Figs. 1 & 2).

As is evident, Fig.1 and Fig. 2, represent the linear and logarithmic numberlines, respectively. A slightly different way to interpret

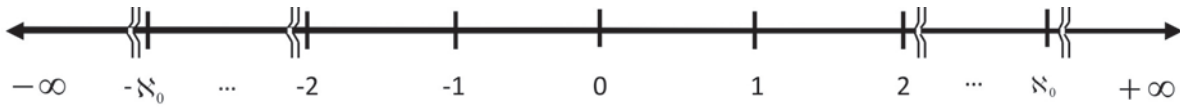


Fig. 1

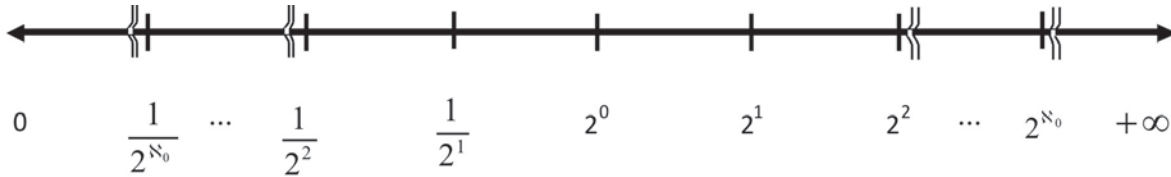


Fig. 2

the logarithmic numberline to base 2 is to say that it is made of the power set of terms on the linear numberline. Thus corresponding to 2 on the linear numberline there is  $2^2$ , its powerset on the logarithmic numberline and corresponding to -1 on the linear numberline we have  $2^{-1} (\frac{1}{2^1})$ , its powerset on the logarithmic numberline. While this notion is easy to understand for terms on the RHS of Fig. 2, the power set of negative integers is not intuitively plain (“In mathematics you don’t understand things. You get used to them”) (Newmann quoted in Zukov, 1984) However, this procedure affords us a way to generate more numberlines and possibly more Approachable Śunyas.

With *Devās* on one side and *Asurās* on the other  
 The Ocean of Milk was churned;  
 First came hissing poison, and later  
 Glories galore and immortality itself...

The next power set-generated numberline is given in Fig. 3

Further, doing power set operation on the terms of Fig. 3 results in the next numberline shown in Fig. 4

The exercise can be continued endlessly in this manner. Let us construct Table. 11 to look at these numberlines more closely.

Column 10 of Table 11 give the list of ‘countable infinity’ in each case. Thus in the case of row 1, we count 1, 2, 3... , and reach  $N_0$  after  $N_0$  steps. Similarly, in case of row 2, we count as 2 ( $2^1$ ), 4 ( $2^2$ ), 8 ( $2^3$ )... and reach  $2^{N_0}$  ( $N_1$ ) – assuming the Continuum Hypothesis – after  $N_0$  steps. In case of row 3, we proceed as 4 ( $2^2$ ), 16 ( $2^4$ ), 256 ( $2^8$ )... and reach  $N_2$  after  $N_0$  steps. Finally, in case of row 4, the counting sequence is

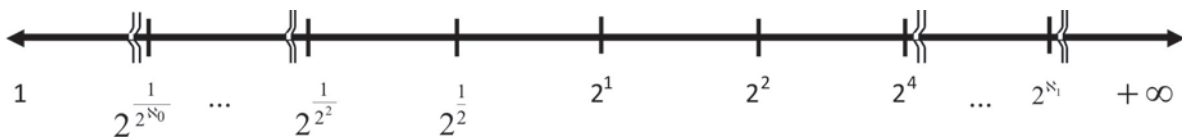


Fig. 3

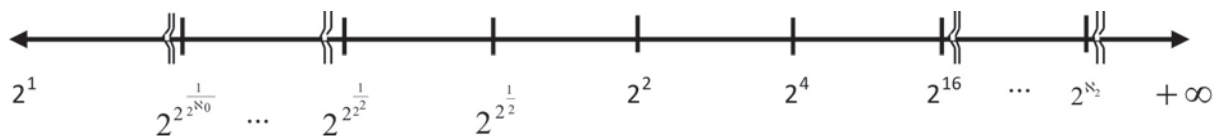


Fig. 4

**Table 11:** Ideal Limits and Reachable Limits of powerset-generated numberlines

Ideal Limit (IL) on the left	Reachable limit (RL) on the left	Few numbers to the left of mid point			Mid point	Few numbers to the right of mid point			Reachable limit (RL) on the right
$-\infty$	$-\aleph_0$	-3	-2	-1	0	1	2	3	$\aleph_0$
0	$\frac{1}{2^{\aleph_0}}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8	$\aleph_1$
1	$2^{\frac{1}{2^{\aleph_0}}}$	$2^{\frac{1}{8}}$	$2^{\frac{1}{4}}$	$2^{\frac{1}{2}}$	2	4	16	256	$\aleph_2$
2	$2^{2^{\frac{1}{2^{\aleph_0}}}}$	$2^{\frac{8}{\sqrt{2}}}$	$2^{4\sqrt{2}}$	$2^{\sqrt{2}}$	4	16	$2^{16}$	$2^{256}$	$\aleph_3$

$2^4, 2^{16}, 2^{256}$  etc. and we reach  $\aleph_3$  after  $\aleph_0$  steps. Such counting with ‘exponential footsteps’ continued in a similar manner will take us to higher and higher cardinals in each case.

With the first step *Vāmana* measured the earthly realm  
 And with the next all other ones;  
 Where was the place for keeping the third? –  
 To honour his word, *Mahābali* offered the sphere of his head...

It is not significant that the base chosen for our exercise is 2. We could as well have chosen base 10. In row 2, we could have counted as  $10 (10^1), 100 (10^2), 1000 (10^3)...$  and reached  $10^{\aleph_0}$  ( $\aleph_1$ ) after  $\aleph_0$  steps. In case of row 3, we could have proceeded as  $10^{10} (10^{10^1}), 10^{100} (10^{10^2}), 10^{1000} (10^{10^3})...$  and reached  $(10^{10^{\aleph_0}})$  after  $\aleph_0$  steps. Finally, in case of row 4, the counting sequence could have been  $10^{10^{10^1}}, 10^{10^{10^2}}, 10^{10^{10^3}}$  etc. and we would have reached  $(10^{10^{10^{\aleph_0}}})$  after  $\aleph_0$  steps.

**8. Intimation of Approachable Śunyas**

Take the interval 0 to 1. Into how many parts can this interval be divided? Start dividing 1 by the successive values on the RHS of row 1,

Table. 11. We will get the following results:  $\frac{1}{1},$

$\frac{1}{2}, \frac{1}{3}$  etc. and reach  $\frac{1}{\aleph_0}$  after  $\aleph_0$  steps. We have reached a value close to 0 but not absolute 0. Addition-based increasing of denominator is ineffective now as  $\frac{1}{\aleph_0 + 1} = \frac{1}{\aleph_0}$ . We can perhaps

call  $\frac{1}{\aleph_0}$  the ‘Zeroth Approachable Śunya’.

Similarly, we can divide 1 into smaller and smaller parts using successive values on the RHS of row 2. The results obtained in this case would be

$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}$ , and we would reach  $\frac{1}{2^{\aleph_0}}$  after  $\aleph_0$  steps.

Again we have reached another value closer to 0 but not absolute 0. Multiplication-based increasing of denominator is ineffective now as

$\frac{1}{2^{\aleph_0}} \times \frac{1}{2} = \frac{1}{2^{\aleph_0+1}} = \frac{1}{2^{\aleph_0}}$ . Incidentally this is the FAS. Changing the base to 10 instead of 2, we

would get the limiting value as  $\frac{1}{10^{\aleph_0}}$ , which in

any case is  $\frac{1}{2^{\aleph_0}}$  only. Assuming Continuum

Hypothesis<sup>3</sup>,  $2^{\aleph_0} = \aleph_1$ , and so  $\frac{1}{2^{\aleph_0}} = \frac{1}{\aleph_1}$ .



**Table 12:** Arithmetic of Zero

Arithmetic of $2^{-\infty} = \frac{1}{2^\infty}$ with finite numbers	Arithmetic of zero with finite numbers	Arithmetic of 0 with itself	Arithmetic of $\aleph_0$ with itself
$n + \frac{1}{2^\infty} = n$	$n + 0 = n$	$0 + 0 = 0$	$\aleph_0 + \aleph_0 = \aleph_0$
$n - \frac{1}{2^\infty} = n$	$n - 0 = n$	$0 - 0 = 0$	$\aleph_0 - \aleph_0 = \aleph_0$
$n \times \frac{1}{2^\infty} = \frac{1}{2^\infty}$	$n \times 0 = 0$	$0 \times 0 = 0$	$\aleph_0 \times \aleph_0 = \aleph_0$
$n \div \frac{1}{2^\infty} = n \times \frac{2^\infty}{1} = 2^\infty$	$\frac{n}{0} = \text{Undefined}$	$= \text{Undefined}$	$\aleph_0 \div \aleph_0 = \text{Undefined}$

We can continue this operation of dividing the interval between 0 and 1 using the successive terms of row 3. (For easier understanding we will use base 10 instead of 2). Now the results would

be  $\frac{1}{10^{10^1}}, \frac{1}{10^{10^2}}, \frac{1}{10^{10^3}}$  etc. After  $\aleph_0$  steps we

would reach the value of  $\frac{1}{10^{10^{\aleph_0}}}$  and there will be

$10^{10^{\aleph_0}}$  such intervals between 0 and 1. Further squaring of this number is ineffective as

$$\frac{1}{10^{10^{\aleph_0+1}}} = \frac{1}{10^{10^{\aleph_0}}}$$

Assuming Continuum Hypothesis,  $10^{10^{\aleph_0}} = 2^{2^{\aleph_0}} = \aleph_2$  and so  $\frac{1}{10^{10^{\aleph_0}}} = \frac{1}{\aleph_2}$ .

This is a smaller value than FAS but it is not the last of the possible Approachable *Śunya*. We can proceed in this manner and come up with smaller and smaller values into which the interval 0 to 1 can be divided. This is one approach to generating smaller and smaller Approachable *Śunya*. Putting

it differently, the values  $\frac{1}{\aleph_0}, \frac{1}{\aleph_1}, \frac{1}{\aleph_2}$ , etc.

(assuming Continuum Hypothesis) could be candidates for the never-ending values of Approachable *Śunyas*. And their defining characteristic would be that these are ‘numbers whose value cannot be further reduced even if they are divided by any finite number’.

“Do not come near here; remove your sandals from your feet, for the place on which you are standing is holy ground.”

(Exodus 3:5)

### 9. Zero as a transfinite cardinal

If  $x$  is a term in row 1 of Table 11, then the corresponding term in row 2 is  $2^x$ . Since IL of LHS of row 1 is  $-\infty$ , we can see that the IL of LHS of second row must be  $2^{-\infty}$ , where  $\infty$  is the Unapproachable Absolute Infinity. But the IL of the LHS of the logarithmic numberline is 0 (Unapproachable Absolute Zero). Therefore, we can say that  $0 = 2^{-\infty}$ . Substituting  $2^{-\infty}$  instead of 0 in the linear numberline, we can see why 0 behaves like a transfinite cardinal in its interactions with finite numbers (n).

As seen in columns 1 and 2 of Table 12, the arithmetic of 0 parallels the arithmetic of  $\frac{1}{2^\infty}$  except in the case of division. As hinted in Section 9 of the earlier paper (Basant & Panda, 2013) and as highlighted in Table. 12, arithmetic of 0 can be better understood if 0 is treated as a transfinite cardinal. Multiplication by 0 ( $\frac{1}{2^\infty}$ ) immediately destroys / modifies the limited cardinality of finite numbers. Thus, we have (3×0)

= (4×0), but it would be more appropriate to state it as (3×0) = (4×0) = 0. In order to clarify division by zero, it is better to represent 0 as  $\frac{1}{2^\infty}$  in that operation.

Thousands of raindrops splash into the stream  
 The river subsumes their individual irks;  
 Mighty rivers rush into the sea  
 The ocean smothers their individual quirks...

**10. Cantor set and the arithmetic of cardinal division**

Cantor set (Byers, 2007) is the set of points remaining in the interval between 0 and 1 after middle thirds have been removed repeatedly from it and its descendant intervals. Thus, in the first instance the interval  $\frac{1}{3}$  to  $\frac{2}{3}$  is removed and in the second instance the middle-third of the two remaining intervals, namely  $\frac{1}{9}$  to  $\frac{2}{9}$  and  $\frac{7}{9}$  to  $\frac{8}{9}$  are removed. After repeating this process  $\aleph_0$  times, what is the length of the remaining interval. This is usually calculated by computing the length of intervals removed. This length is the sum of the convergent geometric series  $\frac{1}{3} + 2\left(\frac{1}{3}\right)^2 + 2^2\left(\frac{1}{3}\right)^3 + 2^3\left(\frac{1}{3}\right)^4 + \dots$  and its *ideal sum* is  $\frac{1}{3} \div \left(1 - \frac{2}{3}\right) = 1$ . Thus the Cantor set is shown to contain zero length and yet, paradoxically, it can be shown to contain  $2^{\aleph_0}$  points in it.

Let us look at the Cantor set in terms of the lengths remaining at the end of  $\aleph_0$  iterations.

At the end of first instance, we have  $\frac{2}{3}$ , which is

further reduced to  $\left(\frac{2}{3}\right)^2$  in the next instance and

$\left(\frac{2}{3}\right)^3$  in the third instance and so on. What will be

the length remaining after  $\aleph_0$  instances?

Obviously the answer is  $\left(\frac{2}{3}\right)^{\aleph_0}$ . What is the value

of  $\left(\frac{2}{3}\right)^{\aleph_0}$ , or, its equivalent  $\frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} \times \dots = \frac{2^{\aleph_0}}{3^{\aleph_0}}$ ?

If the intervals that have been removed add up to

1, then obviously, this value of  $\frac{2^{\aleph_0}}{3^{\aleph_0}}$ , which

represents the measure of the intervals remaining, must be equal to 0. But this obviously is far-fetched

since even  $\frac{1}{3^{\aleph_0}}$  cannot be equal to absolute zero

(0) as shown in various places in Sections: 1 – 5. How to resolve this contradiction?

At the outset it must be pointed out that by the Theorem of FAS, the *actual sum* of a convergent geometric series is less than the ideal

sum by  $\frac{1}{2^{\aleph_0}}$ , so that we can say that the sum of

intervals removed after  $\aleph_0$  steps is not 1 but  $(1 -$

$\frac{1}{2^{\aleph_0}})$ . It is therefore evident that an interval of

$\frac{1}{2^{\aleph_0}}$  will remain after  $\aleph_0$  iterations have been

gone through. For greater clarity let us construct the following table to view the intervals remaining.

Thus it can be seen that after  $\aleph_0$  instances,

there will be  $2^{\aleph_0}$  intervals of length  $\frac{1}{3^{\aleph_0}}$  still



**Table 13:** length and number of remaining intervals

Instance	No. of intervals remaining	Length of each interval remaining
1	2	$\frac{1}{3}$
2	4	$\frac{1}{9}$
3	$2^3$	$\frac{1}{3^3}$
4	$2^4$	$\frac{1}{3^4}$
$\aleph_0$	$2^{\aleph_0}$	$\frac{1}{3^{\aleph_0}}$

remaining in the Cantor Set. This is equivalent to stating that the total interval that will remain in the Cantor Set after  $\aleph_0$  instances will be

$$2^{\aleph_0} \times \frac{1}{3^{\aleph_0}}, \text{ which is equivalent to } \frac{2^{\aleph_0}}{3^{\aleph_0}}.$$

Thus by the Theorem of FAS, we got the remainder left in the Cantor Set as  $\frac{1}{2^{\aleph_0}}$  and by

the reasoning of Table 14 as  $\frac{2^{\aleph_0}}{3^{\aleph_0}}$ . Are these two values equivalent? Let us find out the value of

$\frac{2^{\aleph_0}}{3^{\aleph_0}}$ . To do so we shall invoke the rules of cardinal arithmetic, especially cardinal multiplication. If

$m$  and  $n$  are two infinite cardinals, then  $m \times n = \max \{m, n\}$ . Thus  $\aleph_0 \times 2^{\aleph_0} = 2^{\aleph_0}$ . The same result can be interpreted this way also: in the multiplication of two infinite cardinals, *the*

*'higher' cardinal prevails and the other one is reduced to 1, the multiplicative identity*, so that  $\aleph_0 \times 2^{\aleph_0} = 1 \times 2^{\aleph_0} = 2^{\aleph_0}$ .

Similarly, in the division of two transfinite cardinals also, the 'higher' cardinal prevails and the other one is reduced to 1. Thus  $\frac{\aleph_0}{2^{\aleph_0}} = \frac{1}{2^{\aleph_0}}$ .

Using this interpretation,  $\frac{2^{\aleph_0}}{3^{\aleph_0}} = \frac{1}{3^{\aleph_0}}$ . But

$\frac{1}{3^{\aleph_0}} = \frac{1}{2^{\aleph_0}}$  and so the interval remaining in the

Cantor set after  $\aleph_0$  steps is  $\frac{1}{2^{\aleph_0}}$ . Thus by both calculations, the equivalent length of interval remaining in the Cantor set after  $\aleph_0$  iterations is

$\frac{1}{2^{\aleph_0}}$ . Further it can be noted that this interval of  $\frac{1}{2^{\aleph_0}}$  contains as many points in them ( $2^{\aleph_0}$  as per

the last row of Table 13) as the original interval of  $[0,1]$  (as per Section: 5) from which the Cantor set was generated.

Let us check this method of cardinal division on another example.

$$\frac{4}{2} = 2 \text{ and so } \frac{4}{2} \times \frac{4}{2} \times \frac{4}{2} \times \dots \text{ must be equal}$$

to  $2^{\aleph_0}$ . But  $\frac{4}{2} \times \frac{4}{2} \times \frac{4}{2} \times \dots = \frac{4^{\aleph_0}}{2^{\aleph_0}}$ . How can  $\frac{4^{\aleph_0}}{2^{\aleph_0}}$  be

equal to  $2^{\aleph_0}$ ? By the interpretation of cardinal multiplication and division, the 'smaller cardinal'

$2^{\aleph_0}$  in  $\frac{4^{\aleph_0}}{2^{\aleph_0}}$  is reduced to 1 and so we have

$$\frac{4^{\aleph_0}}{2^{\aleph_0}} = \frac{4^{\aleph_0}}{1} = 4^{\aleph_0} = 2^{\aleph_0}.$$

Just as in the case of multiplication and division the 'smaller' cardinal is reduced to the multiplicative identity (1), in the case of addition and subtraction, the 'smaller' cardinal is *reduced to 0, the additive identity*, so that  $\aleph_0 + 2^{\aleph_0} = 0 + 2^{\aleph_0} = 2^{\aleph_0}$  and  $3^{\aleph_0} + 5^{\aleph_0} = 0 + 5^{\aleph_0} = 5^{\aleph_0}$ . This

interpretation holds even in the case of subtraction so that  $2^{\aleph_0} - \aleph_0 = 2^{\aleph_0} - 0 = 2^{\aleph_0}$ . Also,  $5^{\aleph_0} - 3^{\aleph_0} = 5^{\aleph_0} - 0 = 5^{\aleph_0}$ .

### 11. Pattern and proof

We have shown by interpretation that

$$\frac{2^{\aleph_0}}{3^{\aleph_0}} = \frac{1}{3^{\aleph_0}} = \frac{1}{2^{\aleph_0}}. \text{ If this interpretation is correct,}$$

$$\text{then } \frac{3^{\aleph_0}}{2^{\aleph_0}} \text{ must be equivalent to } \frac{3^{\aleph_0}}{1} = 3^{\aleph_0} = 2^{\aleph_0}.$$

But  $\frac{3^{\aleph_0}}{2^{\aleph_0}}$  is equivalent to  $\frac{3}{2} \times \frac{3}{2} \times \frac{3}{2} \times \dots$  and this is equivalent to  $1.5 \times 1.5 \times 1.5 \times \dots = 1.5^{\aleph_0}$ . Therefore the inevitable conclusion follows that  $1.5^{\aleph_0} = 2^{\aleph_0}$ .

But interpretation is one thing and rigorous proof is another. Is there any other way to *prove* that  $\frac{2^{\aleph_0}}{3^{\aleph_0}} = \frac{1}{2^{\aleph_0}}$  and  $1.5^{\aleph_0} = 2^{\aleph_0}$ ?

We have,  $\sqrt{2} = 1.414\dots$ . Therefore, we can say that

$$\sqrt{2} < 1.5 < 2^{\aleph_0} \quad \dots(12)$$

Raising all to the power  $\aleph_0$ , we have

$$\sqrt{2}^{\aleph_0} < 1.5^{\aleph_0} < (2^{\aleph_0})^{\aleph_0}$$

Now  $\sqrt{2}^{\aleph_0} = \sqrt{2}^{2 \times \aleph_0}$  since  $2 \times \aleph_0 = \aleph_0$ . Further  $(2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0 \times \aleph_0} = 2^{\aleph_0}$ .

So  $\sqrt{2}^{2 \times \aleph_0} < 1.5^{\aleph_0} < 2^{\aleph_0}$ . But  $\sqrt{2}^{2 \times \aleph_0} = ((\sqrt{2})^2)^{\aleph_0} = 2^{\aleph_0}$ .

The value of  $\sqrt{2}^{\aleph_0}$  can be explained in another way also:  $\sqrt{2}^{\aleph_0} = \sqrt{2} \times \sqrt{2} \times \sqrt{2} \times \dots$   
 $= (\sqrt{2} \times \sqrt{2}) \times (\sqrt{2} \times \sqrt{2}) \times (\sqrt{2} \times \sqrt{2}) \times \dots$   
 $= 2 \times 2 \times 2 \times \dots = 2^{\aleph_0}$ .

Therefore, the inequality (12) raised to the power  $\aleph_0$  is equal to  $2^{\aleph_0} < 1.5^{\aleph_0} < 2^{\aleph_0}$ . As the first

and third terms are equal, the middle term also must be equal, (Cantor–Bernstein–Schroeder theorem) (Dauben, 1990) and, so  $1.5^{\aleph_0} = 2^{\aleph_0}$ . We have thus proved few results here. One that  $1.5^{\aleph_0} = 2^{\aleph_0}$ ; another, that  $\sqrt{2}^{\aleph_0} = 2^{\aleph_0}$ . It further implies that if  $x > \sqrt{2}$ , then  $x^{\aleph_0} = 2^{\aleph_0}$ . Therefore  $1.6^{\aleph_0} = 2^{\aleph_0}$ ,  $1.768^{\aleph_0} = 2^{\aleph_0}$  etc.

Doting Yaśodā who fed him motherly milk  
 And crooked Poothana who suckled Him poisoned milk  
 Were both elevated to an exalted plane  
 For they had been touched by the Infinite Being...

Using similar argument it can be shown that  $(\sqrt[3]{2})^{\aleph_0} = 2^{\aleph_0}$ ,  $(\sqrt[4]{2})^{\aleph_0} = 2^{\aleph_0}$  etc. The enabling property in these cases will be that  $3 \times \aleph_0 = \aleph_0$  and  $4 \times \aleph_0 = \aleph_0$  etc. Again, as in the case of  $\sqrt{2}$  stated above,  $(\sqrt[3]{2})^{\aleph_0} = (\sqrt[3]{2} \times \sqrt[3]{2} \times \sqrt[3]{2}) \times (\sqrt[3]{2} \times \sqrt[3]{2} \times \sqrt[3]{2}) \times \dots = 2 \times 2 \times 2 \times \dots = 2^{\aleph_0}$  etc. also.

It must be noted that as we proceed in the manner of  $\sqrt[3]{2}, \sqrt[4]{2}, \sqrt[5]{2}$  etc., the first term of inequality (12) will be getting closer and closer to 1. We can therefore state the following:

**Theorem 1:** *If k is any real number such that  $\sqrt[n]{2} \leq k \leq 2^{\aleph_0}$ , where n is any large but finite natural number such that  $\sqrt[n]{2} \approx 1$ , then  $k^{\aleph_0} = 2^{\aleph_0}$ .*

What is the result of raising any real number between 0 and 1 to  $\aleph_0$  (i.e., the value of  $k^{\aleph_0}$ , where  $0 < k < 1$ )? Let us write a similar inequality like (12). Since  $\frac{1}{\sqrt{2}} = 0.707\dots$

$$\frac{1}{2} < \frac{2}{3} < \frac{1}{\sqrt{2}} \quad \dots(13)$$

Raising all to the power of  $\aleph_0$ , we have

$\left(\frac{1}{2}\right)^{N_0} < \left(\frac{2}{3}\right)^{N_0} < \left(\frac{1}{\sqrt{2}}\right)^{N_0}$ . Since  $N_0 = 2 \times N_0$ , this is equivalent to  $\frac{1}{2^{N_0}} < \left(\frac{2}{3}\right)^{N_0} < \frac{1}{\sqrt{2}^{2 \times N_0}}$ . But  $\sqrt{2}^{2 \times N_0} = 2^{N_0}$  as in inequality (12) and so we have  $\frac{1}{2^{N_0}} < \left(\frac{2}{3}\right)^{N_0} < \frac{1}{2^{N_0}}$ . Therefore  $\left(\frac{2}{3}\right)^{N_0} = \frac{2^{N_0}}{3^{N_0}} = \frac{1}{2^{N_0}}$ , since the first and third terms of the inequality are equal as in the case of (12). Further, we can replace  $\sqrt{2}$  with  $\sqrt[3]{2}$ ,  $\sqrt[4]{2}$  etc. as in inequality (13) and bring the value of the third term closer and closer to 1. And as in the previous case, the enabling property in this case would also be that  $3 \times N_0 = N_0$ ,  $4 \times N_0 = N_0$  etc. We can therefore state the following:

**Theorem 2:** *If k is any real number such that*

$$\frac{1}{2^{N_0}} \leq k \leq \frac{1}{\sqrt[n]{2}}, \text{ where } n \text{ is any large but finite}$$

*natural number such that  $\sqrt[n]{2} \approx 1$ , then  $k^{N_0} = \frac{1}{2^{N_0}}$ .*

Enemies keep vigil at the border  
Narrow is the no-man's land in between;  
Whoever strays into either side  
Is captured by the respective side...

**12. Proof and Understanding**

But proof is one thing and understanding is another. Therefore let us seek to understand what is going on here. Thurton (1994) has remarked with autobiographical candor on the efforts of mathematicians: "...What we are producing is human understanding. We have many different ways to understand and many different processes that contribute to our understanding. We will be more satisfied, more productive and happier if we recognize and focus on this". And William Bayers (2007) has put it emphatically: "Mathematics is about understanding! Proofs are important to the

extent that they help develop an understanding of some mathematical situation".

Proof is the tender mango –  
Bitter is the approach to the seed;  
Understanding is the ripe fruit –  
Aah, the juicy way to the nut...

Why is it that  $1 - 0.999... = \frac{1}{2^{N_0}}$ ? How is it that the real numbers begin with  $\frac{1}{10^{N_0}} = \frac{1}{2^{N_0}}$  and continue as  $\frac{2}{10^{N_0}}$ ,  $\frac{3}{10^{N_0}}$  etc.? How to understand that  $k^{N_0} = \frac{1}{2^{N_0}}$  where  $0 < k < 1$ ?

To understand these in an even simpler manner, let us construct a variation of the Cantor set where we remove the middle  $\frac{8}{10}$  from the available intervals.

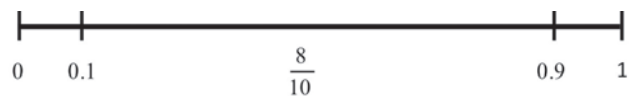


Fig. 5

We start with the interval 0 to 1 and are left with the intervals 0 to 0.1 and 0.9 to 1 after the first instance (Fig. 5). It can be seen that the length of each of these remaining intervals is  $\frac{1}{10}$  of the original interval. We will initially concentrate only on the extreme left (EL) and extreme right (ER) intervals. In the next instance, the intervals on the EL become 0 to 0.01 and the interval on the ER becomes 0.99 to 1. These intervals are obviously  $\frac{1}{10^2}$  of the original interval. Let us look at the Table: 14 for more details and clarity:

It can therefore be seen that  $2^{N_0}$  intervals of length  $\frac{1}{10^{N_0}} = 0.000...1$  will be left in the

**Table 14:** Length of intervals and their number

Instance	EL Interval	Length of EL interval	ER Interval	Length of ER interval	No. of intervals at each instance	Total interval remaining at each instance
1	0 to 0.1	$\frac{1}{10}$	0.9 to 1	$\frac{1}{10}$	2	$\frac{2}{10} = \frac{1}{5} = 0.2$
2	0 to 0.01	$\frac{1}{10^2}$	0.99 to 1	$\frac{1}{10^2}$	$2^2$	$\frac{2^2}{10^2} = \frac{1}{5^2} = (0.2)^2$
3	0 to 0.001	$\frac{1}{10^3}$	0.999 to 1	$\frac{1}{10^3}$	$2^3$	$\frac{2^3}{10^3} = \frac{1}{5^3} = (0.2)^3$
$\aleph_0$	0 to 0.000...1	$\frac{1}{10^{\aleph_0}}$	0.999...9 to 1	$\frac{1}{10^{\aleph_0}}$	$2^{\aleph_0}$	$\frac{2^{\aleph_0}}{10^{\aleph_0}} = \frac{1}{5^{\aleph_0}} = (0.2)^{\aleph_0} = \frac{1}{2^{\aleph_0}}$

eight-by-ten Cantor set after  $\aleph_0$  iterations. Thus in respect to the number of intervals remaining ( $2^{\aleph_0}$ ) and the length of those intervals ( $\frac{1}{10^{\aleph_0}} = \frac{1}{2^{\aleph_0}}$ ) the eight-by-ten Cantor set is equivalent to the one-by-three Cantor set.

Since at each stage we are removing the portion between  $\frac{1}{10}$  and  $\frac{9}{10}$ , the intervals remaining cannot contain any of the digits 2 to 8. Further, as the intervals generated at each instance is  $\frac{1}{10}$  of the previous interval, the decimal representation of the end points of intervals affords us an address of these points.

Let us list out the intervals for the first three iterations.

- a) 1<sup>st</sup> instance: (0.0, 0.1), (0.9,1.0)
- b) 2<sup>nd</sup> instance: (0.00, 0.01), (0.09,0.10), (0.90, 0.91), (0.99,1.00)
- c) 3<sup>rd</sup> instance: (0.000, 0.001), (0.009, 0.010), (0.090, 0.091), (0.099, 0.100), (0.900, 0.901), (0.909, 0.910), (0.990, 0.991), (0.999, 1.000)

Let us take any one point, say 0.901. As the first digit after decimal point is 9, in the first

instance this point is on the right side (between 0.9 and 1.0). The second digit after decimal point is 0. So this number in the second instance is on the left hand side (between 0.90 and 0.91) of the interval generated. In the third instance the digit is 1 and this says that the point is on the right hand side of the next generated interval (between 0.900 and 0.901). Using this property it is possible to list all the numbers at each instance as decimal representations.

However, a more intuitive approach would be to represent these numbers as rational numbers as given below:

- a) 1<sup>st</sup> instance:  $(\frac{0}{10}, \frac{1}{10}), (\frac{9}{10}, \frac{10}{10})$
- b) 2<sup>nd</sup> instance:  $(\frac{0}{10^2}, \frac{1}{10^2}), (\frac{9}{10^2}, \frac{10}{10^2}), (\frac{90}{10^2}, \frac{91}{10^2}), (\frac{99}{10^2}, \frac{100}{10^2})$
- c) 3<sup>rd</sup> instance:  $(\frac{0}{10^3}, \frac{1}{10^3}), (\frac{9}{10^3}, \frac{10}{10^3}), (\frac{90}{10^3}, \frac{91}{10^3}), (\frac{99}{10^3}, \frac{100}{10^3}), (\frac{900}{10^3}, \frac{901}{10^3}), (\frac{909}{10^3}, \frac{910}{10^3}), (\frac{990}{10^3}, \frac{991}{10^3}), (\frac{999}{10^3}, \frac{1000}{10^3})$

Looking at the above, we can observe the following.

- a) In all instances, the initial nominators remain unchanged. Only, more nominators are added as instances increase.
- b) The denominators increase as  $10^1, 10^2, 10^3$  etc. and the number of intervals increase as  $2^1, 2^2, 2^3$  etc.
- c) The nominators do not contain digits 2 to 8. Another way of saying this is that after digit 1 must come 9 as the corresponding digit in the next nominator. Further we can see that nominators increase by addition. For iteration 3, the increasing nominators are obtained by addition in the following sequence: 1, 8, 1, 80, 1, 8, 1, 800, 1, 8, 1, 80, 1, 8, 1. This affords us an algorithmic way to generate the nominators:

Algorithm:

Step 1: start with 0

Step 2: add 1 to get the next nominator

Step 3: convert digit 1 (as obtained in Step 2) to 9 to get next nominator

Step 4: repeat steps 2 and 3

Step 5: end when the nominator is equal to denominator

With the help of the above steps, we can generate few of the points representing the intervals on the

LHS after  $\aleph_0$  instances as below:  $(\frac{0}{10^{\aleph_0}}, \frac{1}{10^{\aleph_0}}),$   
 $(\frac{9}{10^{\aleph_0}}, \frac{10}{10^{\aleph_0}}), (\frac{90}{10^{\aleph_0}}, \frac{91}{10^{\aleph_0}}), (\frac{99}{10^{\aleph_0}}, \frac{100}{10^{\aleph_0}}),$   
 $(\frac{900}{10^{\aleph_0}}, \frac{901}{10^{\aleph_0}}), (\frac{909}{10^{\aleph_0}}, \frac{910}{10^{\aleph_0}}), (\frac{990}{10^{\aleph_0}}, \frac{991}{10^{\aleph_0}}),$   
 $(\frac{999}{10^{\aleph_0}}, \frac{1000}{10^{\aleph_0}})$  etc.

These are the intervals remaining on the left hand side. With simple reversal of the algorithmic steps given above, we can list the points on the RHS also:

$$(\frac{100\dots0}{10^{\aleph_0}}, \frac{999\dots999}{10^{\aleph_0}}), (\frac{999\dots991}{10^{\aleph_0}}, \frac{999\dots990}{10^{\aleph_0}}),$$

$$(\frac{999\dots910}{10^{\aleph_0}}, \frac{999\dots909}{10^{\aleph_0}}), (\frac{999\dots901}{10^{\aleph_0}}, \frac{999\dots900}{10^{\aleph_0}}),$$

$$(\frac{999\dots100}{10^{\aleph_0}}, \frac{999\dots099}{10^{\aleph_0}}), (\frac{999\dots091}{10^{\aleph_0}}, \frac{999\dots090}{10^{\aleph_0}}),$$

$$(\frac{999\dots010}{10^{\aleph_0}}, \frac{999\dots009}{10^{\aleph_0}}), (\frac{999\dots001}{10^{\aleph_0}}, \frac{999\dots000}{10^{\aleph_0}})$$

etc.

Here we can notice that all the  $2^{\aleph_0}$  intervals are  $\frac{1}{10^{\aleph_0}}$  long and these range from  $(\frac{0}{10^{\aleph_0}}, \frac{1}{10^{\aleph_0}})$  on the left to  $(\frac{999\dots9}{10^{\aleph_0}}, \frac{100\dots0}{10^{\aleph_0}})$  on the right. Thus we can understand that  $0.999\dots(\frac{999\dots9}{10^{\aleph_0}})$  is different from  $1(\frac{100\dots0}{10^{\aleph_0}})$  just as  $0(\frac{0}{10^{\aleph_0}})$  is different from  $0.000\dots1(\frac{1}{10^{\aleph_0}})$ .

Further, we can easily infer that if the  $\frac{8}{10}$  intervals had not been removed from the original interval of 0 to 1, then the points remaining in it would have been  $\frac{0}{10^{\aleph_0}}, \frac{1}{10^{\aleph_0}}, \frac{2}{10^{\aleph_0}}, \frac{3}{10^{\aleph_0}}, \frac{4}{10^{\aleph_0}}$  etc. up to  $\frac{100\dots0}{10^{\aleph_0}}$ , where  $100\dots0$  is a number  $\aleph_0$

digits long. But these are the  $10^{\aleph_0} = 2^{\aleph_0}$  real numbers from 0 and 1 as listed in Table 10 and as explained in the subsequent paragraphs of Section: 5. Thus by combinatorial argument and by the eight-by-ten Cantor set analysis, we get the same list of real numbers for the interval 0 to 1. Further it can be seen from Column 7 of Table 14 that the total interval remaining at the end of instances is  $\frac{2^{\aleph_0}}{10^{\aleph_0}} = \frac{2}{10} \times \frac{2}{10} \times \frac{2}{10} \dots = \frac{1}{5^{\aleph_0}} = \frac{1}{2^{\aleph_0}}$ . Again, from the same Column 7,  $\frac{2^{\aleph_0}}{10^{\aleph_0}} = \frac{2}{10} \times \frac{2}{10} \times \frac{2}{10} \dots =$

$$(0.2)^{\aleph_0} = \frac{1}{2^{\aleph_0}} \text{ (by Theorem 2)}$$



We started with the interval 0 to 1 and produced  $2^{\aleph_0}$  intervals of  $\frac{1}{10^{\aleph_0}}$  in  $\aleph_0$  iterations. What if we had started with an interval of, say 0.0001 ( $\frac{1}{10^4}$ )?  $\frac{1}{10^4}$  is the length of EL interval in row 4 of Table 14 and so the above question is equivalent to asking: How many rows intervene between 4 and  $\aleph_0$ ? This obviously is  $\aleph_0 - 4 = \aleph_0$ . Therefore  $\aleph_0$  iterations have to be gone through before the interval 0.0001 is reduced to  $\frac{1}{10^{\aleph_0}}$ . And when  $\aleph_0$  iterations have been gone through,  $2^{\aleph_0}$  intervals will be generated. Thus whether we start with the interval of 1 or 0.0001, there will be  $2^{\aleph_0}$  intervals of  $\frac{1}{10^{\aleph_0}}$  remaining in the end (Compare this to the permutation-based argument at the end of Section: 5). We can start with any small interval  $\frac{1}{10^n}$ , where  $n$  is any finite natural number, and it will contain the same number of intervals ( $2^{\aleph_0}$ ) as the interval 0 to 1 because  $\aleph_0 - n = \aleph_0$ .

We can even start with an interval of 10 (0 to 10). From this point of view, the interval 0 to 1 is the left hand portion of the first iteration of the interval 0 to 10. After that the remaining iterations will go on as in Table. 15. Thus the total number of iterations gone through will be  $\aleph_0 + 1 = \aleph_0$ . So the interval 0 to 10 will contain as many intervals as the interval 0 to 1 after  $\aleph_0$  iterations. Further,  $\aleph_0 + \aleph_0 = \aleph_0$ . Therefore, if we begin with the interval 0 to  $10^{\aleph_0}$  also, we will be left with  $2^{\aleph_0}$  intervals of  $\frac{1}{10^{\aleph_0}}$  after  $\aleph_0$  iterations. But as

mentioned in Section: 6,  $10^{\aleph_0} = 2^{\aleph_0}$  is the upper limit of the countable numbers with place value notation. And so we can see that whether we start with the whole of the numberline or a tiny length of it and subject it to eight-by-ten Cantor set iteration, we will get  $2^{\aleph_0}$  intervals of  $\frac{1}{10^{\aleph_0}}$  in  $\aleph_0$  steps.

To understand the generation of real numbers in an even easier manner, we can construct another variation of the Cantor set. This time we divide the interval 0 to 1 into 10 equal parts and remove the alternate parts such that the remaining intervals are as shown in Fig.6.

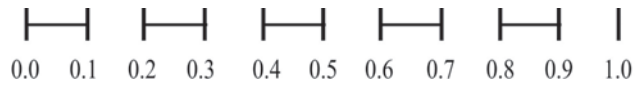


Fig. 6

Each of the 5 remaining intervals is also subjected to division by 10 and removal of alternate parts. The outcome of this exercise after such iterations, is given in Table. 15.

It is easy to see that intervals remaining in the five-by-ten Cantor set can be derived similar to the intervals remaining in the eight-by-ten Cantor set. These are,

- a) 1<sup>st</sup> instance:  $(\frac{0}{10}, \frac{1}{10}), (\frac{2}{10}, \frac{3}{10}), (\frac{4}{10}, \frac{5}{10}), (\frac{6}{10}, \frac{7}{10}), (\frac{8}{10}, \frac{9}{10})$ .
- b) 2<sup>nd</sup> instance:  $(\frac{0}{10^2}, \frac{1}{10^2}), (\frac{2}{10^2}, \frac{3}{10^2}), (\frac{4}{10^2}, \frac{5}{10^2}), (\frac{6}{10^2}, \frac{7}{10^2}), (\frac{8}{10^2}, \frac{9}{10^2})$  etc.
- c)  $\aleph_0^{\text{th}}$  (w) instance:  $(\frac{0}{10^{\aleph_0}}, \frac{1}{10^{\aleph_0}}), (\frac{2}{10^{\aleph_0}}, \frac{3}{10^{\aleph_0}}), (\frac{4}{10^{\aleph_0}}, \frac{5}{10^{\aleph_0}}), (\frac{6}{10^{\aleph_0}}, \frac{7}{10^{\aleph_0}}), (\frac{8}{10^{\aleph_0}}, \frac{9}{10^{\aleph_0}})$  and so on.

**Table 15:** Length and Number of intervals remaining in Five-by-ten Cantor set

Instance	EL Interval	Length of EL interval	No. of intervals remaining at each instance	Total length of interval remaining at each instance (column 3 × 4)
1	0 to 0.1	$\frac{1}{10}$	5	$\frac{5}{10} = \frac{1}{2}$
2	0 to 0.01	$\frac{1}{10^2}$	$5^2$	$\frac{5^2}{10^2} = \frac{1}{4}$
3	0 to 0.001	$\frac{1}{10^3}$	$5^3$	$\frac{5^3}{10^3} = \frac{1}{2^3}$
$\aleph_0$	0 to 0.000...1	$\frac{1}{10^{\aleph_0}}$	$5^{\aleph_0} = 2^{\aleph_0}$	$\frac{5^{\aleph_0}}{10^{\aleph_0}} = \frac{1}{2^{\aleph_0}}$

But the points representing the intervals, as obtained at the  $\aleph_0^{\text{th}}$  ( $\omega$ ) instance, are the real numbers as given in column 3 of Table 10.

Incidentally, this five-by-ten Cantor set is a graphic illustration of the paradox of Zeno. We start with interval 0 to 1, then remove  $\frac{5}{10} = \frac{1}{2}$  of it. Now the remaining portion is  $1 - \frac{5}{10} = \frac{5}{10}$ . Next we remove

$\frac{5}{10}$  of this remaining portion as  $\frac{5}{10} \times \frac{5}{10} = \frac{5^2}{10^2} = \frac{1}{2^2}$ .

In the next instance the portion removed is  $\frac{5^3}{10^3} = \frac{1}{2^3}$  of the original interval and so on up to  $\frac{5^{\aleph_0}}{10^{\aleph_0}} = \frac{1}{2^{\aleph_0}}$ . The sum of parts so removed adds up

to  $\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots + \frac{1}{2^{\aleph_0}} = 1$ , ideally, though by the Theorem of FAS, the actual sum of the series is  $1 - \frac{1}{2^{\aleph_0}}$ . As the last row of Table.15 shows, even after the removal of the ideal sum of

1, there will be  $5^{\aleph_0} = 2^{\aleph_0}$  intervals of length  $\frac{1}{10^{\aleph_0}}$  still remaining in the original interval (0 to 1) and

these will be  $(\frac{0}{10^{\aleph_0}}, \frac{1}{10^{\aleph_0}})$ ,  $(\frac{2}{10^{\aleph_0}}, \frac{3}{10^{\aleph_0}})$ ,  $(\frac{4}{10^{\aleph_0}}, \frac{5}{10^{\aleph_0}})$ ,  $(\frac{6}{10^{\aleph_0}}, \frac{7}{10^{\aleph_0}})$ ,  $(\frac{8}{10^{\aleph_0}}, \frac{9}{10^{\aleph_0}})$ ,  $(\frac{20}{10^{\aleph_0}}, \frac{21}{10^{\aleph_0}})$ ,  $(\frac{22}{10^{\aleph_0}}, \frac{23}{10^{\aleph_0}})$  etc. with a total length of  $5^{\aleph_0} \times \frac{1}{10^{\aleph_0}} = \frac{5^{\aleph_0}}{10^{\aleph_0}} = (\frac{5}{10})^{\aleph_0} = (0.5)^{\aleph_0} = \frac{1}{2^{\aleph_0}}$  (by Theorem 2).

Telescope aids as eye-tight tool  
To observe the distant and faint  
But acts as self-worn blinkers  
Against obvious lateral truths...

The mansion of mathematics is built  
Using granite blocks of rigour;  
Its foundation is axiomatically anchored  
In molten magma of faith...

**13. Theorems: 1 & 2 and the Continuum Hypothesis**

Simply put, the Continuum Hypothesis asserts that there is no infinite cardinal  $p$  such that  $\aleph_0 < p < 2^{\aleph_0}$ . Cantor (Byers, 2007) believed in it but could not resolve it either way – whether it was true or false. It was Kurt Godel (Byers, 2007) who showed in 1940 that the hypothesis cannot be disproved in the standard axiomatic set theory.



In 1963 Paul Cohen (Byers, 2007) showed that the hypothesis could not be proved from the same set of axioms either. So the Continuum Hypothesis is generally held to be independent of the axioms of set theory.

However, using cardinal arithmetic, we can explore whether any infinite cardinal  $p$  can possibly exist between  $\aleph_0$  and  $2^{\aleph_0}$ . Addition cannot be a tool to reach  $p$  since it cannot cross  $\aleph_0$  as  $\aleph_0 + 1 = \aleph_0$  and  $\aleph_0 + \aleph_0 = \aleph_0$ . Similarly multiplication (repeated addition) also cannot reach  $p$  since it too cannot cross  $\aleph_0$  as  $\aleph_0 \times 2 = \aleph_0$  and  $\aleph_0 \times \aleph_0 = \aleph_0$ . The other operation capable of taking us across  $\aleph_0$  is exponentiation (repeated multiplication). Here we may note that 2 raised to any finite number  $b$  cannot produce an infinite cardinal since  $2^b$  can only be finite. When  $b$  becomes  $\aleph_0$ , the smallest infinite cardinal, then we get  $2^{\aleph_0} > p$ .

We would therefore be justified in thinking that perhaps a real number less than 2 when raised to the power of  $\aleph_0$  would produce the infinite cardinal  $p$ . However Theorems 1 and 2 disprove this. Any real number between 1 and  $2^{\aleph_0}$ , when raised to  $\aleph_0$  is equal to  $2^{\aleph_0}$  and any real number between 0 and 1, when raised to  $\aleph_0$  is equal to  $\frac{1}{2^{\aleph_0}}$ . And  $1^{\aleph_0}$  is equal to 1 only. Therefore, our efforts to go past  $\aleph_0$  using exponentiation of real numbers between 1 and 2 would only result in  $2^{\aleph_0}$  and *nothing short of that*.

Counting stars in the Milky Way  
 The spacefarer journeys forth;  
 Other stars are in other galaxies  
 There are no stars in between...

Thus Theorems 1 and 2 bolster the hunch of Cantor that there cannot be any infinite cardinal  $p$  such that  $\aleph_0 < p < 2^{\aleph_0}$ .

We can try tetration (repeated exponentiation). Though this operation will take us beyond and is therefore not pertinent to the question at hand, it could have a bearing on the generalized continuum hypothesis.  $2 \wedge 2 \wedge 2 \wedge \dots = 2^{(2 \times 2 \times 2 \times \dots)} = 2^{2^{\aleph_0}} > 2^{\aleph_0} \neq p$ . Also,  $\sqrt[n]{2} \wedge \sqrt[n]{2} \wedge \sqrt[n]{2} \wedge \dots = \sqrt[n]{2^{(\sqrt[n]{2} \times \sqrt[n]{2} \times \sqrt[n]{2} \times \dots)}} = \sqrt[n]{2^{(2 \times 2 \times 2 \times \dots)}} = \sqrt[n]{2^{2^{\aleph_0}}} = (\sqrt[n]{2})^{2 \times 2^{\aleph_0}} = 2^{2^{\aleph_0}}$ .

We can therefore state the following inequality

$$\sqrt[n]{2} \leq k \leq 2^{\aleph_0} \dots(14)$$

where  $n$  is any large but finite natural number such that  $\sqrt[n]{2} \approx 1$  and  $k$  is some real number.

We will now raise all the three terms to the power of  $2^{\aleph_0}$ . Incidentally, raising any number  $k > 1$  to the power of  $2^{\aleph_0}$  is equivalent to tetration of that number since  $k^{2^{\aleph_0}} = k^{k^{\aleph_0}} = k^{(k \times k \times k \times \dots)}$ . Also, raising any number  $k$  to the power of  $\aleph_0$  is equivalent to exponentiation or repeated multiplication of that number as  $k^{\aleph_0} = k \times k \times k \times \dots$ . Thus we have  $(\sqrt[n]{2})^{2^{\aleph_0}} \leq k^{2^{\aleph_0}} \leq (2^{\aleph_0})^{2^{\aleph_0}}$ . As  $n \times 2^{\aleph_0} = 2^{\aleph_0}$ , where  $n$  is any large but finite natural number, we have

$$\left(2^{\frac{1}{n}}\right)^{n \times 2^{\aleph_0}} \leq k^{2^{\aleph_0}} \leq (2^{\aleph_0})^{2^{\aleph_0}} \quad \text{Since}$$

$\frac{1}{n} \times n \times 2^{\aleph_0} = 2^{\aleph_0}$  and  $(2^{\aleph_0})^{2^{\aleph_0}} = 2^{\aleph_0 \times 2^{\aleph_0}} = 2^{2^{\aleph_0}}$ , we have  $2^{2^{\aleph_0}} \leq k^{2^{\aleph_0}} \leq 2^{2^{\aleph_0}}$ . As the first and third terms are equal, the middle term must also be equal and so  $k^{2^{\aleph_0}} = 2^{2^{\aleph_0}}$ .

**Theorem 3:** If  $k$  is any real number such that  $\sqrt[n]{2} \leq k \leq 2^{N_0}$ , where  $n$  is any large but finite natural number such that  $\sqrt[n]{2} \approx 1$ , then  $k^{2^{N_0}} = 2^{2^{N_0}}$ .

Similarly, select  $k$  such that  $\frac{1}{2^{N_0}} \leq k \leq \frac{1}{\sqrt[n]{2}}$   
 ... (15)

Raising all to  $2^{N_0}$ ,  $\frac{1}{(2^{N_0})^{2^{N_0}}} \leq k^{2^{N_0}} \leq \frac{1}{(\sqrt[n]{2})^{2^{N_0}}}$ . As

in the case of derivation of Theorem 3,  $(2^{N_0})^{2^{N_0}} = 2^{2^{N_0}}$  and  $n \times 2^{N_0} = 2^{N_0}$ , where  $n$  is any large but finite natural number. Therefore the

present inequality reduces to  $\frac{1}{2^{2^{N_0}}} \leq k^{2^{N_0}} \leq \frac{1}{2^{2^{N_0}}}$ .

Again, as the first and last terms are equal, the middle term of the inequality too must be equal and so we have:

**Theorem 4:** If  $k$  is any real number such that

$\frac{1}{2^{N_0}} \leq k \leq \frac{1}{\sqrt[n]{2}}$ , where  $n$  is any large but finite

natural number such that  $\sqrt[n]{2} \approx 1$ , then  $k^{2^{N_0}} = \frac{1}{2^{2^{N_0}}}$ .

Theorems 3 and 4 show that tetration of any number between 1 and 2 will only produce  $2^{2^{N_0}}$  and not any infinite cardinal between  $2^{N_n}$  and  $2^{2^{N_0}}$ . Deriving such theorems for higher and higher infinite cardinals should not be difficult and such results would strengthen the claim of generalized continuum hypothesis that between one infinite cardinal  $\aleph_n$  and its powerset, there cannot be any infinite cardinal  $q$  such that  $\aleph_n < q < 2^{\aleph_n}$ .

Ants carry rice in bits and grains;  
 There are no containers in their world...  
 Farmers carry their harvest in bundles;

Few spilled grains are not reckoned by them...

Factory counts its production in tons;  
 There are no half-filled bags in its tally...  
 Ship carries its cargo in containers;  
 There are no odd-size boxes in its world...

**14. The quantum world of numbers**

At various places in this paper it has been shown that FAS cannot be discarded as nothing. In fact it has been shown to be of significance in many calculations. But at various places in the earlier paper it was contented that the FAS could be *defined as* a kind of zero. Aren't these two stands opposite and contradictory?

It must be mentioned that we are entering the quantum world of numbers where the contradictory must be accepted as complementary aspects of the same reality. Wave and particle are contradictory at the level of sea and sand but complementary at the level of photons and electrons. Similarly, Sunyas can be either nothing or a thing of value depending on the context.

Add a spoonful of salt to its licking tongue  
 And the ocean ignores your labored affront;  
 Add a spoonful more salt to his favourite dish  
 And the husband is likely to froth and fume...

When the zeroth sunya is a measurable quantity  
 The First Approachable *Sunya* is a mere nothing;  
 When the natural numbers are taken into reckoning  
 All the *Sunyas* are reduced to nothing...

When the race-track circumference is an exact measure  
 The diameter is a hazy string of decimals;  
 When the centre-fixing diameter is an exact measure  
 The circumference is an endless decimal that is closed...

Bohr: Not one move of the nectar-thief  
bee  
Shall be known in advance;  
Einstein: Not one flower in the garden  
Shall be left unpollinated...

Jostling like motley crowd at carnival  
Digits of golden ratio stretch forth;  
Filing out like pilgrims from a shrine  
Digits of its continued fraction descend...

Not one step of the Butter-Thief Child  
Could be known in advance;  
Yet not one household of the Gopis  
Was bereft of the thrill of His pranks...

### 15. More numberlines and Approachable *Śunyas*

It is interesting to note from Table. 11 that power set of negative numbers are rational numbers and power set of rational numbers are irrational numbers.

Another interesting fact about Table. 11 is that each row has a distinctive operation that produces the terms of that numberline. Thus, given one term of a numberline, we can produce the other terms by this operation. In the case of linear numberline this is addition. Given any number on this numberline, we can produce the numbers to the right and left of it by addition / subtraction. Similarly, on the logarithmic numberline, the distinctive operation is multiplication / division and for irrational numbers it is exponentiation (square / square root) and so on.

Teachers deliver course content  
In sedate linear scale;  
Students burn the midnight oil  
At turbocharged logarithmic scale...

His opponent walked the distance  
With silent measured steps;  
Zeno wove his paradoxes  
In leaping logarithmic scale;

Draupadi dreaded with trepidation  
One sari minus one sari is zero;  
Dussasana realized with perspiration  
Infinity minus infinity is still infinity...

However many numberlines are generated by the powerset operation, there will always be some difference between RL and IL on both RHS and LHS of Table. 11. Could the difference between RL and IL in the LHS (Column I – column 2) be a clue to more Approachable Sunyas?

In the derivation of Theorem 2, we used the terms  $\sqrt[3]{2}$ ,  $\sqrt[4]{2}$ ,  $\sqrt[5]{2}$  etc. to get to the smallest value close to 1 that can be reached by subjecting 2 to the repeated operation of exponentiation in the sequence  $2^{\frac{1}{2}}$ ,  $2^{\frac{1}{3}}$ ,  $2^{\frac{1}{4}}$  ... . The limit of this is obviously  $\sqrt[{\aleph_0}]{2}$  or  $2^{\frac{1}{\aleph_0}}$ . Now,  $2^{\frac{1}{\aleph_0}}$  is a value very close to 1 but it is not the closest. We could have carried out the exponentiation of 2 in the sequence  $2^{\frac{1}{2}}$ ,  $2^{\frac{1}{2^2}}$ ,  $2^{\frac{1}{2^4}}$  ... ,  $2^{\frac{1}{2^{\aleph_0}}}$  also. This is the same as taking the square root of 2, then taking the square root of that answer and continuing in that manner.

Also, it can be seen that  $2^{\frac{1}{2}}$ ,  $2^{\frac{1}{2^2}}$ ,  $2^{\frac{1}{2^4}}$  ... ,  $2^{\frac{1}{2^{\aleph_0}}}$ , are the LHS of Row 3 of Table 11 with its ideal limit as 1. Assuming the Continuum Hypothesis,  $2^{\aleph_0} = \aleph_1$  and so  $2^{\frac{1}{2^{\aleph_0}}} = 2^{\frac{1}{\aleph_1}}$ . We have now derived two values ( $2^{\frac{1}{\aleph_0}}$  and  $2^{\frac{1}{\aleph_1}}$ ) which are close to 1, but not equal to it. So the difference of these values from 1 could be also considered as Approachable Sunyas. And thus we have the first two of these

new Sunyas as  $(2^{\frac{1}{\aleph_0}} - 1)$  and  $(2^{\frac{1}{\aleph_1}} - 1)$ . It may be

noted that as explained in Section: 8,  $\frac{1}{\aleph_0}$ ,  $\frac{1}{\aleph_1}$ ,  $\frac{1}{\aleph_2}$  etc. are candidates for Approachable Sunyas of the first order.

So the second order of Approachable *Śunyas* would consist of raising 2 to the value of the first order Approachable *Śunyas* and subtracting 1 from it.

**Table 16:** Approachable Sunyas of first and second order

First order Approachable <i>Śunyas</i>	Deriving principle	Second order Approachable <i>Śunyas</i>	Deriving principle
$\frac{1}{\aleph_0} - 0 = \frac{1}{\aleph_0}$	Division	$2^{\frac{1}{\aleph_0}} - 1$	Exponentiation
$\frac{1}{\aleph_1} - 0 = \frac{1}{\aleph_1}$		$2^{\frac{1}{\aleph_1}} - 1$	
$\frac{1}{\aleph_2} - 0 = \frac{1}{\aleph_2}$		$2^{\frac{1}{\aleph_2}} - 1$	
$\frac{1}{\aleph_n} - 0 = \frac{1}{\aleph_n}$		$2^{\frac{1}{\aleph_n}} - 1$	

We can therefore list the first and second order Approachable Sunyas (Table 16):

Can there be third order Approachable *Śunyas*? And a fourth, and so on...

Just as there are endless transfinite cardinals on the way to Unapproachable Absolute Infinity, there will be endless Approachable *Śunyas* on the way to Unapproachable Absolute Zero. "Hitherto shalt thou come, but no further..." (Job 38:11). Nothing, is impossible!

Whatever, the length of rope  
That Yaśodā could gather with  
determination,  
The same just was not sufficient  
To go around the tiny waist of Child  
Kṛṣṇa...

### Acknowledgement

I must be candid: anyone can – with His Grace this one did.

(Themes for many of the verses in this paper and the previous one (Basant & Panda, 2013) have been taken from Maharshi Veda-Vyasa's *Bhagavatha Purana*, a book that delineates the interaction between finite and Infinite.)

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- (Some of the verses appearing in this paper are from my next book of poems tentatively titled, 'Writ On Water'. Others were written for the occasion.)