Novel Proofs for Summations in the *Nisṛṣṭārthadūtī*

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(Received 05 January 2017)

Abstract

Recent scholarly investigations have brought to light various mathematical proofs given by the astronomers and mathematicians of the Kerala school. However, not much is known regarding the proofs discussed by mathematicians outside this school. The present paper aims to partially fill this gap by presenting the interesting proofs of various summations discussed by Munīśvara—a 17th century astronomer from Vārāṇasī—in his *Nisṛṣṭārthadūtī*, a hitherto unpublished commentary on Bhāskara's *Līlāvatī*.

Key words: *Līlāvatī*, Munīśvara, *Nisṛṣṭārthadūtī*, Proofs, *Saṅkalita*, Summations, *Upapatti*

1. INTRODUCTION

Starting with Āryabhaṭa, Indian mathematicians have dealt with a wide range of topics ranging from basic arithmetic to advanced geometry and algebra in their works. Among the many topics, most works include a chapter describing summations and progressions, which gives relations for the sum of first n natural numbers, their squares, cubes etc., apart from various rules pertaining to arithmetic and geometric progressions. However, no primary text of Indian mathematics presents the proofs of these relations—a task which is left to the commentators.

Earlier studies have shown that the Kerala school of mathematics has contributed greatly towards various proofs of summation relations given in texts like *Āryabhaṭīya* and *Līlāvatī*. For instance, Mallayya has discussed[1] a variety of geometric proofs for summation relations given by Nīlakaṇṭha and Śaṅkara in their texts *Āryabhaṭīya-bhāṣya* (a commentary on *Āryabhaṭīya*) and *Kriyākramakarī* (a commentary on *Līlāvatī*) respectively. Similarly, Sarma et al. (2008) show that Jyeṣṭhadeva in his *Gaṇita-yukti-bhāṣā* discusses some ingenious proofs in connection with the infinite series expression for π and other trigonometric functions given by Mādhava. Such extensive and perceptive proofs as discussed by these authors are not generally known to have been presented by mathematicians outside of the Kerala school.

However, either based on a preliminary survey confined only to source works, or being carried away by certain remarks of other scholars, the general opinion that has been prevalent among historians of mathematics is one of absence of proofs in the Indian mathematical tradition. Some careful historians who concede the presence of proofs, are of the view that this tradition of presenting

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[1] See Mallayya 2001 and Mallayya 2002.

proofs commences more or less with the advent and flourishing of the Kerala school. Interestingly, Bronkhorst (2001) having cited an important paper of M. D. Srinivas presenting a list of texts that deal with proofs in an appendix, commenting on the tradition of providing mathematical *upapattis* in India, has noted[2] that "all of them date from the 16th and 17th centuries". In contrast to this, M. D. Srinivas has shown[3] that the earliest exposition of *upapattis* in Indian mathematical and astronomical works dates back at least to the time of Govindasvāmin (800 CE). In any case, a study of Munīśvara's *Nisṛṣṭārthadūtī* (c. 17th century)—an unpublished commentary[4] on *Līlāvatī*—has unveiled some interesting proofs for various summation relations described in the *Līlāvatī*. The present paper brings to light these novel and insightful proofs discussed by Munīśvara.

Towards this end, we first give a brief background of Munīśvara and his *Nisṛṣṭārthadūtī* in Section 2. Next, as a precursor to discussing Munīśvara's proofs for summation relations given in *Līlāvatī*, we briefly discuss the relevant verses from this text in Section 3. Then, we provide an overview of the proofs given by other authors in Section 4. Finally, we discuss the proofs described in the *Nisṛṣṭārthadūtī* in Section 5, and conclude the discussion with our remarks in Section 6.

2. MUNĪŚVARA AND HIS *NISṚṢṬĀRTHADŪTĪ*

Munīśvara, also known as Viśvarūpa,[5] was the son of the illustrious Raṅganātha[6]—the author of the famous *Gūḍhaprakāśa* commentary of the *Sūryasiddhānta*. Born in Śaka 1525 (1603 CE), Munīśvara resided in Vārāṇasī, and was an astronomer and mathematician of great repute. He is known to have composed several works—both original as well as commentaries on important treatises—the details of which are summarised in Table 1.

Owing to its popularity, the *Līlāvatī* has attracted a large number of commentaries in Sanskrit as well as other languages, several of which are well regarded. Some of the important Sanskrit commentaries of *Līlāvatī* include Gaṅgādhara's *Gaṇitāmṛtasāgarī*, Sūryadāsa's *Gaṇitāmṛtakūpikā*, Gaṇeśa Daivajña's *Buddhivilāsinī*, Raṅganātha's *Mitabhāṣiṇī*, Narāyaṇa's *Karmapradīpikā*, and Śaṅkara's *Kriyākramakarī*. Among these commentaries by various eminent authors, Paṇḍit Sudhākara Dvivedī (c. 19th century) pays glorious tribute to the *Nisṛṣṭārthadūtī* of Munīśvara by describing it as the best among the excellent (*uttamottamā*):[7]

लीलावतीटीका च सम्प्रति सर्वत्र भारतवर्षेऽन्यटीकात उ-
त्तमोत्तमेति ज्योतिर्विद्भिर्मता ।

[2]Bronkhorst 2001, p. 65. The paper cited by Bronkhorst—Srinivas (1990)—is a reproduction of Srinivas (1987). Among the 17 texts listed in Appendix 1 by Srinivas in his paper, at least three of them are prior to 16th century. Notwithstanding this, the assertion ("all of them") made by Bronkhorst only obfuscates the truth and adds to the prevailing confusion. For a more detailed list of texts dealing with proofs, see Srinivas (2005).

[3]Srinivas 2005, pp. 214–217.

[4]The authors obtained copies of two manuscripts of *Nisṛṣṭārthadūtī* from the archives of the Sarasvati Bhavana Library, which is currently under the auspices of the Sampurnanand Sanskrit University, Vārāṇasī.

[5]Munīśvara refers to his alternate name in the following verse (see Dvivedi 1933, p. 93) of his work *Marīci*—a commentary on the *Siddhāntaśiromaṇi*:

> मुनीश्वरापराख्येन विश्वरूपेण घृष्यते ।
> बुद्धिशाणे मरीच्यर्थं तत्सिद्धान्तशिरोमणिः ॥
> That crown jewel of *Siddhāntas* (*Siddhāntaśiromaṇi*) is ground on the whetstone of intellect by Viśvarūpa, generally known by the name Munīśvara, for [making it brilliantly reflect] the rays of light (*marīcyartham*).

The beautiful simile employed to explain the name *Marīci* for the commentary is indeed worth noting here.

[6]Dvivedi 1933, pp. 91–92.

[7]Dvivedi 1933, p. 91.

Table 1. Works composed by Munīśvara.

No.	Name of the work	Brief description
1	*Siddhāntasārvabhauma*	Full-fledged *siddhānta* text
2	*Svāśayaprakāśinī*	Auto-commentary on the *Siddhāntasārvabhauma*
3	*Nisṛṣṭārthadūtī*	Commentary on *Līlāvatī*
4	*Marīci*	Commentary on *Siddhāntaśiromaṇi*
5	*Pāṭīsāra*	Text presenting the essence of arithmetic

> In the present times, the commentary on *Līlāvatī* [by Munīśvara] is considered to be the best among the excellent [commentaries] by astronomers and mathematicians throughout India.

The word *Nisṛṣṭārthadūtī* can be translated as "emissary of the bestowed[8] meaning". Anyone who reads this commentary cannot but agree that this emissary not only conveys the intended message, but adds more value in a number of ways. That is, besides explaining the basic import of Bhāskara's verses, Munīśvara further adds proofs of various results given by Bhāskara, highlights the proofs given by earlier scholars (critiquing them if necessary), explains the grammatical peculiarities of the words employed in *Līlāvatī*, and so on. In short, through his commentary, he tries to address most of the doubts that may arise in the minds of its readers. In this sense, the title *Nisṛṣṭārthadūtī* is very apt for the commentary. Indeed, while outlining the significance of the title, Munīśvara himself observes towards the end of the work:[9]

> अयं लीलावत्याः पटुदुरवगाहोऽतिगहनो
> मनोभावो भूयांस्तदधिगतये यां[10] व्यरचयत् ।
> मुनीशस्तामेतां कृतिमकृतिदुष्प्राप्यविषयां
> निसृष्टार्थां दूतीमिव भजत भावेन चतुराः ॥[11]

> The thoughts in the mind of *Līlāvatī* are manifold (*bhūyān*), quite deep (*atigahana*), and difficult to comprehend (*duravagāha*) even for the smart ones. In order to facilitate [a clear] understanding of that, this work (*kṛti*), which Munīśvara has composed like an emissary (*dūtī*) of the betowed meaning, O prudent ones, gracefully grasp that (*dūtī*) who is not available (*duṣprāpya*) for the immeritorious (*akṛti*).

While the above verse is quite beautiful in its own right, Munīśvara also appears to intend it as a tribute to the poetic genius of Bhāskara by the use of phrases like *atigahana*, *manobhāvo bhūyān* etc. It is also remarkable that Munīśvara takes forward the poetic flourish employed in the last verse of *Līlāvatī*, which reads:

> येषां सुजातिगुणवर्गविभूषिताङ्गी
> शुद्धाखिलव्यवहृतिः खलु कण्ठसक्ता ।
> लीलावतीह सरसोक्तिमुदाहरन्ती
> तेषां सदैव सुखसंपदुपैति वृद्धिम् ॥२७२॥

> Here [in this world], those for whom the *Līlāvatī*—whose sections (*aṅga*) are adorned with procedures for reductions of fractions (*jāti*), rules for multiplication (*guṇa*), squaring (*varga*) [etc.], which has descriptions that are faultless, [and] which presents elegant and enchanting examples—is memorised, for them the wealth of happiness will indeed always increase.

[8] By bestowed, we mean the deeper meaning and nuances intended by the poet in the text.

[9] See Munīśvara 1879, p. 405. Also see Dvivedi 1933, p. 92.

[10] Dvivedi (1933) as well as the manuscripts gives the reading भूयांस्तदधिगतयेमां. However, this appears to be a transcribing error, and the given reading is more appropriate.

[11] The prose order (*anvaya*) is as follows: लीलावत्याः मनोभावः अयं भूयान् अतिगहनः पटुदुरवगाहः । तदधिगतये यां कृतिं निसृष्टार्थां दूतीमिव मुनीशो व्यरचयत् [हे] चतुराः ताम् एताम् अकृतिदुष्प्राप्यविषयां भावेन भजत ।

Bhāskara employs a poetic flourish known as *śleṣālaṅkāra* (pun) in the above verse through which, in addition to the above interpretation, a completely different interpretation is also possible:

> Here [in this world], a beautiful woman who—is high-born and adorned with many virtues, having pure and blemishless conduct, [and] who utters enticing words—is in the embrace of whosoever, their wealth of happiness will indeed always increase.

Whereas the term *Līlāvatī* in the first interpretation refers to the text, in the second interpretation it alludes to a beautiful woman. Munīśvara appears to take forward this brilliant use of pun by Bhāskara in his verse above, where in the same vein, it is possible to interpret *Līlāvatī* once again as a woman, and the *Nisṛṣṭārthadūtī* as the emissary who delivers her message.

3. SUMMATIONS IN THE *LĪLĀVATĪ*

In his *Līlāvatī*, Bhāskara discusses various rules with regards to summations and progression is a chapter titled *Śreḍhīvyavahāra*. In this section, we present the relevant verses[12] dealing with summations from this chapter, along with mathematical notes.

3.1. Summation and sum of sums of first n natural numbers

> सैकपदघ्नपदार्धमथैका-
> द्यङ्कयुतिः किल सङ्कलिताख्या ।
> सा द्वियुतेन पदेन विनिघ्नी
> स्यात्त्रिहृता खलु सङ्कलितैक्यम् ॥११७॥

> Now, the sum of the numbers starting with one is called *saṅkalita*, which is indeed half the number of terms (*pada*) [in the series] multiplied by the *pada* added by one. That [sum] multiplied by the *pada* [which is] added by two, [and] divided by three would indeed be the sum of the *saṅkalitas*.

The above verse, composed in the *Dodhaka* metre, gives the relations for (i) *Saṅkalita*: The summation of the first n integers starting with the number one, and (ii) *Saṅkalitaikya*: The sum of the *saṅkalitas*. Here, the term *pada* refers to the total number of terms (n). Denoting the above two sums as S_n (sum of integers), and V_n (sum of sums) respectively, the relations given in the verse can be expressed using modern mathematical notation as follows:

$$
\begin{aligned}
S_n &= \sum_{i=1}^{n} i = 1 + 2 + \cdots + n \\
&= \frac{n}{2} \times (n+1), \qquad (1) \\
V_n &= \sum_{i=1}^{n} S_i = S_1 + S_2 + \cdots + S_n \\
&= \frac{S_n \times (n+2)}{3}. \qquad (2)
\end{aligned}
$$

3.2. Summation of squares and cubes of natural numbers

> द्विघ्नपदं कुयुतं त्रिविभक्तं
> सङ्कलितेन हतं कृतियोगः ।
> सङ्कलितस्य कृतेस्सममेका-
> द्यङ्कघनैक्यमुदाहृतमाद्यैः ॥११९॥

> Twice the number of terms (*pada*) added by one, divided by three [and] multiplied by *saṅkalita* is the sum of squares (*kṛti*) [of natural numbers]. The sum of cubes of the numbers starting from one has been stated to be equal to the square of *saṅkalita* by the ancestors.

This verse, again in the *Dodhaka* metre, states the rules for the sum of the squares and sum of cubes of a sequence of integers starting with the number one. Let S_{n^2} and S_{n^3} designate the sum of squares and sum of cubes respectively. Then, as

[12] Accepting the reading given in Āpte 1937a as the standard reading.

per the prescription given in the verse

$$S_{n^2} = \sum_{i=1}^{n} i^2 = 1^2 + 2^2 + \cdots + n^2 = \frac{2n+1}{3} \times S_n, \quad (3)$$

$$S_{n^3} = \sum_{i=1}^{n} i^3 = 1^3 + 2^3 + \cdots + n^3 = S_n^2. \quad (4)$$

Before discussing the proofs for the above relations given in *Nisṛṣṭārthadūtī*, we briefly summarise some of the proofs given by earlier mathematicians in the following section.

4. PROOFS OF SUMMATIONS IN EARLIER WORKS

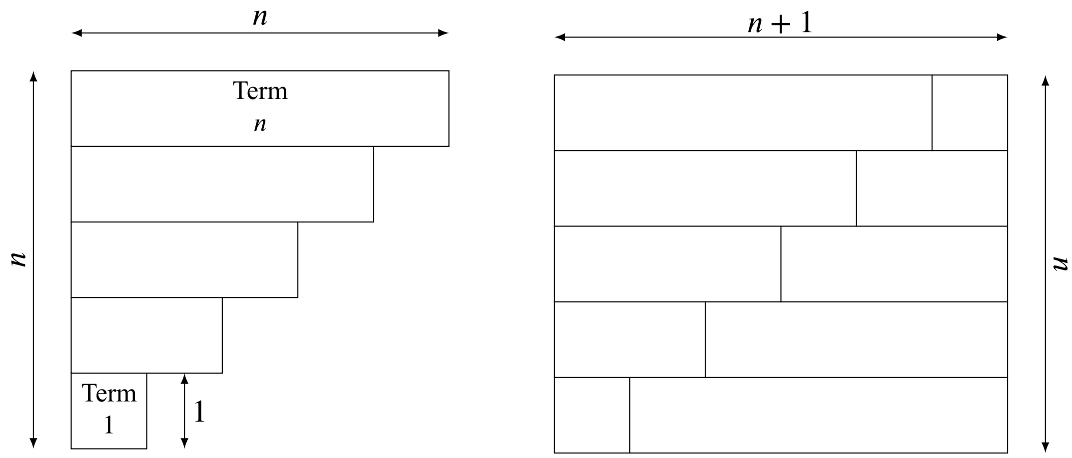
It is well known that in the Indian tradition, the source works of mathematics and astronomy focus only on presenting the rules in a succinct manner, and do not delineate their proofs in the interest of brevity (*lāghava*). Thus, the authors of treatises such as *Āryabhaṭīya* (5th century), *Brahmasphuṭasiddhānta* (7th century), and *Līlāvatī* (12th century), do not present proofs for the mathematical relations given in their texts. This task is typically taken up by the commentators who elaborate, expand, and demonstrate the rules given in the source texts.

Perhaps the first (surviving) text to provide complete proofs for summation relations is Nīlakaṇṭha's *Āryabhaṭīya-bhāṣya* (c. 16th century). Here, Nīlakaṇṭha provides detailed geometric proofs[13] for the various summation relations like the sum of natural numbers, the sum of sums, the sum of squares, and the sum of cubes. For instance, to prove (1), Nīlakaṇṭha visualises the various terms of the sequence as rectangles of width equal to unity, and length equal to the value of the term. As Mallayya (2001) shows, these rectangles are then stacked one upon another to form a *śreḍhīkṣetra* as depicted in Fig. 1a. This *śreḍhīkṣetra* is then joined with another similar but inverted *śreḍhīkṣetra* to form a rectangle as shown in Fig. 1b. The area of this rectangle is then obviously equal to $n \times (n + 1)$. Therefore, the area of one *śreḍhīkṣetra*, which represents the sum of the terms equals $\frac{n(n+1)}{2}$.

Mallayya further discusses how Nīlakaṇṭha expands this technique to provide geometric proofs for other summation relations. He also discusses[14] how Śaṅkara presents similar geometric proofs in his *Kriyākramakarī*. Jyeṣṭhadeva in his *Gaṇita-yukti-bhāṣā*[15] presents an interesting technique for finding the large n behaviour for the summation of all the powers of natural numbers. We do not discuss this here as it is out of the scope of the paper. Interested readers may refer to *Gaṇita-yukti-bhāṣā*, as well as Divakaran's discussion[16] on recursive methods employed in Indian mathematics.

It may be noted that all the three astronomers mentioned above—Nīlakaṇṭha, Śaṅkara, and Jyeṣṭhadeva—belong to the Kerala school founded by Mādhava. Among commentators not belonging to the Kerala school, we find that Gaṇeśa gives a terse proof for (1) in his *Buddhivilāsinī*,[17] a commentary on *Līlāvatī*. He first correctly identifies the middle term of the sequence as the mean of the first and last terms $\left(\frac{n+1}{2}\right)$. Then, he argues that successive numbers on either side of the middle term are lesser or greater than the mean by an equal amount. He then concludes that the summation of the terms is equal to the product of the mean with the total

[13] See Mallayya 2001.
[14] See Mallayya 2002.
[15] See Sarma et al. 2008, pp. 192–197.
[16] Divakaran 2010.
[17] Āpte 1937a, p. 112.



(a) A single *śreḍhīkṣetra*.

(b) Two joined *śreḍhīkṣetras*.

Fig. 1. Nīlakaṇṭha's geometric proof for the sum of integers.

number of terms:[18]

$$S_n = \frac{n(n+1)}{2},$$

which is the same as (1). However, strictly speaking, this argument is valid only if the series has an odd number of terms.

The proofs provided by other astronomers and mathematicians can be gleaned from the citations or references made by Colebrooke. For instance, he credits Kamalākara (contemporary of Munīśvara) with a proof for the sum of integers as follows:[19]

> Kamalākara is quoted by Raṅganātha[20] for a demonstration grounded on placing the numbers of the series in the reversed order under the direct one and adding the two series.

As may be noted from the next section, a similar proof has been presented by Munīśvara too.

5. PROOFS IN *NISṚṢṬĀRTHADŪTĪ*

In this section we discuss the proofs presented in *Nisṛṣṭārthadūtī* for various kinds of summation relations. The study is based on two manuscripts[21] available with us. While the proofs have been presented in neat and succinct Sanskrit prose, there are gaps in certain places which need to be filled with the help of other manuscripts that need to be procured from other sources. Therefore, in this paper we quote only sparingly from the text to give a flavour of Munīśvara's crisp and clear style of writing. However, we completely discuss the proofs given by him using modern mathematical notation for the convenience of readers.

5.1. Sum of natural numbers

Munīśvara presents three proofs for the well known relation (1) which gives the sum of first n natural numbers. We discuss these proofs below.

[18]More explicitly, the additive and the subtractive quantities nullify each other, and each term can be considered equal to the mean. Therefore, the result.

[19]Colebrooke 1967, p. 66.

[20]Confusingly, Munīśvara's father, and Kamalākara's younger brother share the same name Raṅganātha. Paṇḍit Sudhākara Dvivedī notes (Dvivedi 1933, p. 92) of an academic rivalry between Munīśvara and Kamalākara. Therefore, the Raṅganātha in Colebrooke's quote can be assumed to be Kamalākara's brother, the author of the *Mitabhāṣiṇī*. Colebrooke confirms this in his introduction. See Colebrooke 1817, pp. xxvi–xxvii.

[21]Munīśvara 1779 and Munīśvara 1879, both from the Sarasvati Bhavana Library, Varanasi.

Proof 1

The first proof presented by Munīśvara is exactly identical with what is commonly found in mathematics textbooks today. The prescription goes as:[22]

> एकादिपदपर्यन्ताङ्कान् क्रमेण संस्थाप्य तन्मध्ये व्यस्तक्रमेणैकादयो युज्यन्ते । तदा प्रत्येकं सैकपदतुल्याङ्काः स्युः । तेषां योगे सैकपदेन गुणितं पदं स्यात् । अत्र सङ्कलितस्य द्विगुणतया पर्यवसानादेतदर्धं सैकपदतुल्याङ्कानां सङ्कलितमुपपन्नम् ।
>
> Having placed the numbers beginning from one and ending with the last term (*pada*) sequentially, the numbers one etc. are added to them in reverse order. Then each one [i.e. sum of corresponding terms] would be equal to last term plus one (*saikapada*). When added, the sum would be the last term multiplied by last term plus one. Since the result happens to be twice the summation (*saṅkalita*), it is [indeed] proved that half of this is the [required] summation.

It can be seen that the above prescription is quite concise, and at the same time as lucid as one can expect. As already noted, Munīśvara's contemporary and rival Kamalākara has also given the same proof. How old this proof actually is, is perhaps anybody's guess. In any case, when expressed using standard modern notation the above prescription translates to:

$$S_n = 1 + 2 + \cdots + (n-1) + n.$$

Reversing the sequence of terms in the RHS of the above equation, we have

$$S_n = n + (n-1) + \cdots + 2 + 1.$$

Summing term by term the numbers appearing in the RHS of the above equations, we get

$$2S_n = (n+1) + (n+1) + \cdots + (n+1).$$

Since there are n terms, the sum of the terms in the RHS will be $n(n+1)$. Hence,

$$S_n = \frac{n(n+1)}{2}.$$

Proof 2

The second proof is similar to Gaṇeśa's argument we have alluded to in the previous section. Munīśvara's argument can be paraphrased as follows:[23]

> The first number is one, and the last number is *pada*. Half of their sum is the middle number. The amount of increment up to the last term (from the mid-term) is the same as the amount of decrement until the first term (from the mid-term). Thus, the middle number multiplied by the number of terms would be the sum of the numbers one etc. increasing by one till the last number.

It can be noted that the above argument—like Gaṇeśa's—faces the drawback of assuming an odd number of terms. This is again not surprising as Munīśvara hews closely to Gaṇeśa's *Buddhivilāsinī* (an earlier commentary of *Līlāvatī*) at many places in the *Nisṛṣṭārthadūtī*. Despite its limitations, Munīśvara might have also presented this proof along with the others partly because it is quite simple and elegant, and also because he wanted to record the proof prevalent in the tradition in one place with the rest.

Proof 3

Munīśvara further credits[24] the following third proof to a certain Lakṣmīdāsa. As per the prescription given here, we first need to place the *pada* n in n places, and add the quantities. Thus we have

$$n + n + \cdots + n = n^2.$$

[22] Munīśvara 1879, p. 86.

[23] Munīśvara 1879, pp. 86–87.

[24] Munīśvara 1879, p. 87. Paṇḍit Sudhākara Dvivedī refers to an astronomer-mathematician Lakṣmīdāsa (c. 15th century CE) in his *Gaṇakataraṅgiṇī*, and credits this author with a commentary on the *Līlāvatī* among other works. Perhaps Munīśvara is referring here to this same Lakṣmīdāsa. See Dvivedi 1933, pp. 55–56.

	n^2	$=$	n	$+$	n	$+$	$\ldots$	$+$	n	$+$	n	$+$	n
$-$	S_{n-1}	$=$	$n-1$	$+$	$n-2$	$+$	$\ldots$	$+$	2	$+$	1		
		$=$	1	$+$	2	$+$	$\ldots$	$+$	$n-2$	$+$	$n-1$	$+$	n

Fig. 2. Determining S_n as the difference of n^2 and S_{n-1} in the third proof.

Now, starting with the penultimate quantity on the right-hand side and moving leftwards, we need to successively subtract the numbers 1 to $n-1$. This operation is shown in Fig. 2.

It may be noted that the quantities subtracted add up to S_{n-1}, whereas the resultant quantities add up to S_n. Therefore, we have a general recursive relation[25] connecting two successive summations to n^2:

$$S_n + S_{n-1} = n^2. \tag{5}$$

Now, adding n to both sides of (5), we have

$$S_n + S_{n-1} + n = n^2 + n.$$

As $S_{n-1} + n = S_n$, the above equation reduces to

$$2S_n = n^2 + n$$

$$\text{or,} \quad S_n = \frac{n(n+1)}{2}.$$

Evidently, this approach to the problem is quite instructive as it involves a recursive relation, which helps in arriving at other important results.

5.2. Sum of sums of natural numbers

Munīśvara gives a detailed proof for the derivation of (2) with the help of Fig. 3a.[26] A slightly modified and expanded version of this figure is depicted in Fig. 3b for greater clarity.

Essentially, this figure shows various sequences (the first row representing S_1, the second S_2, and so on, with the last row representing S_n), the sum of whose sums we seek to find. As can be seen from Fig. 3b, the number 1 occurs n times in the first column, whereas each subsequent number occurs one time less than the immediately preceding number from the second column onwards. Therefore, the sum of sums, denoted by V_n, can be expressed as:

$$\begin{aligned} V_n &= n \times 1 + (n-1) \times 2 + (n-2) \times 3 + \\ &\quad \cdots + (n-(n-1)) \times n \\ &= n \times (1 + 2 + \cdots + n) - [1 \times 2 + \\ &\quad 2 \times 3 + 3 \times 4 + \cdots + (n-1) \times n] \\ &= nS_n - 2 \times \left(1 + 3 + \cdots + \frac{(n-1)n}{2}\right) \\ &= nS_n - 2 \times (S_1 + S_2 + \cdots + S_{n-1}) \\ &= nS_n - 2 \times V_{n-1}. \end{aligned}$$

Since $V_{n-1} = V_n - S_n$, we get

$$V_n = (n+2)S_n - 2V_n.$$

Hence,

$$V_n = \frac{S_n(n+2)}{3} = \frac{n(n+1)(n+2)}{1 \cdot 2 \cdot 3}, \tag{6}$$

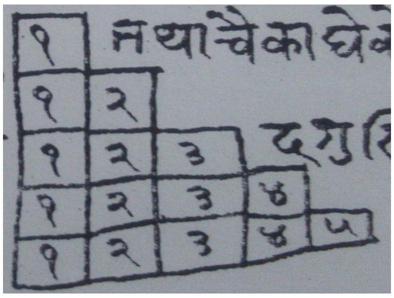
which is the same as (2).

5.3. Sum of squares of natural numbers

Munīśvara presents two proofs for (3), crediting one to Lakṣmīdāsa, and the other to a certain Rāmacandra.

[25] Jyeṣṭhadeva makes use of this same recursive relation in his *Gaṇita-yukti-bhāṣā* for obtaining the approximation $S_n \approx \frac{n^2}{2}$ for large n. See Sarma et al. 2008, p. 193.

[26] For proof and figure, see Munīśvara 1779, pp. 87–88.



(a) Figure given in the manuscript.

$$
\begin{array}{llllll}
\hline
1 & & & & & \\
1 & 2 & & & & \\
1 & 2 & 3 & & & \\
1 & 2 & 3 & 4 & & \\
1 & 2 & 3 & 4 & 5 & \\
\vdots & & & & & \\
1 & 2 & 3 & 4 & 5 & \ldots \; n \\
\hline
\end{array}
$$

(b) Redrawn figure.

Fig. 3. Deriving the relation for the sum of sums.

Proof 1

We first present the passage that describes the proof for the sum of squares of natural numbers. It may be noted that the passage commences with the statement that this *upapatti* is due to Lakṣmīdāsa:[27]

> अत्रोपपत्तिस्तु लक्ष्मीदासोक्ता । सङ्कलितपदघाते पदपर्यन्तमेकादिजातवर्गाणामैक्यस्य रूपोनपदसङ्कलितैक्यस्य च योगोऽवश्यं भवति । कुत एवमिति चेच्छृणु । एकादिपदपर्यन्तानामङ्कानां योगः सङ्कलितम् । तस्मिन् पदगुणिते पदगुणितानामङ्कानां वा योगे समानत्वात् अन्तिमाङ्कः पदगुणितः पदवर्गः । ततः पदेन उपान्तिमाङ्को रूपोनपदमितो गुणितः । स रूपोनपदस्य वर्गो रूपोनपदेन एकगुणेन युतो भवति । पदस्य रूपतदूनपदात्मकखण्डद्वययोगात्मकत्वेन गुणकत्वाभ्युपगमात् । एवं व्युत्क्रमेण तृतीयादयोऽङ्काः[28] पदगुणिताः सन्तो द्व्याद्यूनपदानां वर्गाद्या द्व्यूनपदैर्द्व्यादिगुणितैर्युक्ता भवन्ति, उक्तरीत्या द्व्याद्यूनपदद्व्यादिखण्डद्वययोगात्मकपदस्य गुणकत्वात् । एतेषां गुणितानां योगे एकाद्येकोत्तराङ्कानां वर्गयोगो, रूपोनपदपर्यन्तम् एकाद्येकोत्तराङ्कानां व्यस्तानां क्रमस्थैः एकाद्येकोत्तराङ्कैर्गुणितानां योगेन रूपोनपदस्य प्रागुक्तनिर्णीतसङ्कलितैक्यात्मकेन युतो भवति । तथा च सङ्कलितपदघाते रूपोनपदस्य सङ्कलितैक्योने कृते वर्गयोगः फलितः ।
>
> Here is the *upapatti* as delinated by Lakṣmīdāsa. Indeed, in the product of the summation and *pada* lies the sum of the squares of one and so on till the *pada*, and the *saṅkalitaikya* of *pada*-minus-one. If you ask why it is so, listen [I explain]. The sum of the numbers from one to *pada* is the *saṅkalita*. When that [*saṅkalita*] is multiplied by *pada*, which is the same as the sum of the *pada* multiplied numbers, the last number multiplied by *pada* is the square of the *pada*. Then, [it may be noted that] the penultimate number which is equal to *pada*-minus-one is multiplied by the *pada*. That is [equal to] the sum of the square of *pada*–minus–one and *pada*-minus-one multiplied by one. This is because, the multiplier *pada* [here] is conceived to be the sum of two parts in the form of one and *pada*-minus-one. In a similar manner, the third and subsequent terms in reverse order multiplied by *pada*, are essentially [equal to] the sum of squares of *pada*-minus-two, and *pada*-minus-two multiplied by two etc., since the multiplier [*pada*] is the sum of two parts namely *pada*-minus-two and two etc. as mentioned earlier. [Thus] in the sum of these products, [we find] the sum of sqaures of natural numbers (*ekādyekottarāṅka*) added by—the sum of the products of reversely placed natural numbers from one upto *pada*-minus-one and the natural numbers in order which is the *saṅkalitaikya* of *pada*-minus-one which has been determined and stated earlier. Thus, when the product of *saṅkalita* and *pada* is diminished by the *saṅkali-*

[27] Munīśvara 1879, pp. 92–94.

[28] In the place of तृतीयादयोऽङ्काः which we find in Munīśvara 1779, we find ततो द्वादयोऽङ्काः in Munīśvara 1879. This seems to be a scribal error in the latter.

> *taikya* of *pada*–minus–one terms, what results is sum of squares [of natural numbers upto *pada*].

The procedure outlined above can be expressed in our mathematical notation as follows:

$$\begin{aligned} nS_n &= n \times [1 + 2 + \cdots + (n-1) + n] \\ &= \sum_{i=1}^{n} n \cdot i. \end{aligned}$$

Now, the last term of the above expansion is equal to n^2, while the penultimate term is $n(n-1)$. By rewriting the multiplier (*guṇaka*) n as

$$n = (n-1) + 1,$$

the penultimate term reduces to

$$\begin{aligned} n \cdot (n-1) &= [(n-1) + 1] \cdot (n-1) \\ &= (n-1)^2 + 1 \cdot (n-1). \end{aligned}$$

Similarly, rewriting n as

$$n = (n-2) + 2,$$

the third-last term becomes

$$\begin{aligned} n \cdot (n-2) &= [(n-2) + 2] \cdot (n-2) \\ &= (n-2)^2 + 2 \cdot (n-2). \end{aligned}$$

Therefore, in general, rewriting

$$n = (n-i) + i,$$

we have

$$\begin{aligned} nS_n &= \sum_{i=1}^{n} [(n-i) + i] \cdot i \\ &= \sum_{i=1}^{n} i^2 + \sum_{i=1}^{n} i \cdot (n-i). \end{aligned}$$

It is easily seen that the first term in the RHS of the above equation is the sum of the squares of natural numbers, and the second term is the sum of sums of $n-1$ terms. Hence, using the notation we have employed, the above equation reduces to:[29]

$$nS_n = S_{n^2} + V_{n-1}. \tag{7}$$

From (6), it is obvious that

$$V_{n-1} = \frac{(n-1)\,n\,(n+1)}{1 \cdot 2 \cdot 3}. \tag{8}$$

Also, from (1) we have $S_n = \frac{n(n+1)}{2}$. Using this and (8) in (7), we get

$$\begin{aligned} S_{n^2} &= n \times \frac{n(n+1)}{2} - \frac{(n-1)\,n\,(n+1)}{1 \cdot 2 \cdot 3} \\ &= \frac{n(n+1)}{2} \times \frac{2n+1}{3}, \end{aligned} \tag{9}$$

which is the same as the result given in (3).

Proof 2

Munīśvara also gives a second proof,[30] crediting Rāmacandra for the same. While the proof given by Lakśmīdāsa—as discussed in the previous section—is more algebraically oriented, the one given by Rāmacandra is more visually oriented. According to Rāmacandra, the squares of the first few integers can be depicted visually as shown in Fig. 4a.[31] For greater clarity and generalisation, an expanded version of the table is shown in Fig. 4c.[32] In these two figures, all the columns except for the first column have empty cells. Fig. 4b depicts the same table as in Fig. 4a, without the first column, and with all the empty cells filled with the same number that occupies

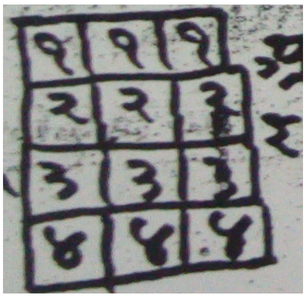
[29] Jyeṣṭhadeva makes use of this relation in his *Gaṇita-yukti-bhāṣā* for obtaining the approximation $S_{n^2} = \frac{n^3}{3}$ for large n. See Sarma et al. 2008, p. 194.

[30] After describing this proof, Munīśvara also expresses some reservations with regard to some finer details in the proof. As this is a moot point, we do not get into the details of it here. This will be taken up when we bring out the edition of the text at a later date. See Munīśvara 1879, pp. 94–95.

[31] That is, the numbers in the first row add up to 1^2, those in the second row add up to 2^2 and so on.

[32] Here the number n occurs n times in the nth row. Their sum adds up to n^2.

(a) Original table in manuscript.



(b) Filled table without first column.

1					
2	2				
3	3	3			
$\vdots$	$\vdots$	$\vdots$			
$n-1$	$n-1$	$n-1$	$\ldots$	$n-1$	
n	n	n	$\ldots$	n	n

(c) Redrawn table.

1	1	$\ldots$	1	1
2	2	$\ldots$	2	2
3	3	$\ldots$	3	3
$\vdots$	$\vdots$	$\ldots$	$\vdots$	$\vdots$
$n-1$	$n-1$	$\ldots$	$n-1$	$n-1$
n	n	$\ldots$	n	n

(d) Filled table without first column.

Fig. 4. Rāmacandra's visual proof for deriving the sum of squares.

the first cell of their respective rows. These filled in numbers are depicted in grey cells in Fig. 4d. By considering the elements column-wise in the greyed portion of Fig. 4d, it can be easily seen that the sum of all numbers in the coloured region (originally empty) turns out to be

$$S_1 + S_2 + \cdots + S_{n-1} = V_{n-1}.$$

As the sum of each column of numbers considering both the coloured and uncoloured cells together is equal to S_n, the sum of the all the uncoloured elements in the grid may be expressed as $(n-1)S_n - V_{n-1}$. Adding the numbers of the first column from Fig. 4c, we have

$$S_{n^2} = S_n + (n-1)S_n - V_{n-1}. \tag{10}$$

Now, from (6) we have

$$\begin{aligned} V_{n-1} &= \frac{(n-1)\, n\, (n+1)}{1 \cdot 2 \cdot 3} \\ &= \frac{(n-1)S_n}{3}. \end{aligned} \tag{11}$$

Using (11) in (10), we get

$$\begin{aligned} S_{n^2} &= S_n + (n-1)S_n - \frac{(n-1)S_n}{3} \\ &= \frac{2n+1}{3} \times S_n. \end{aligned}$$

Fig. 5 shows how the text makes use of a notation system while discussing some intermediary steps in connection with the above proof. Here, the notations प, सं, पव, and पघ stand for *pada*, *saṅkalita*, *pada-varga*, and *pada-ghana*, which represent n, S_n, n^2, and n^3 in our notation system. Accordingly, the underlined terms in the second and third rows in the figure are equivalent to

$$S_n = \frac{n^2+n}{2},$$

and

$$V_n = \frac{n^3+3n^2+2n}{6}.$$

It may be noted that the number appearing below the last term is the denominator for the entire expression.

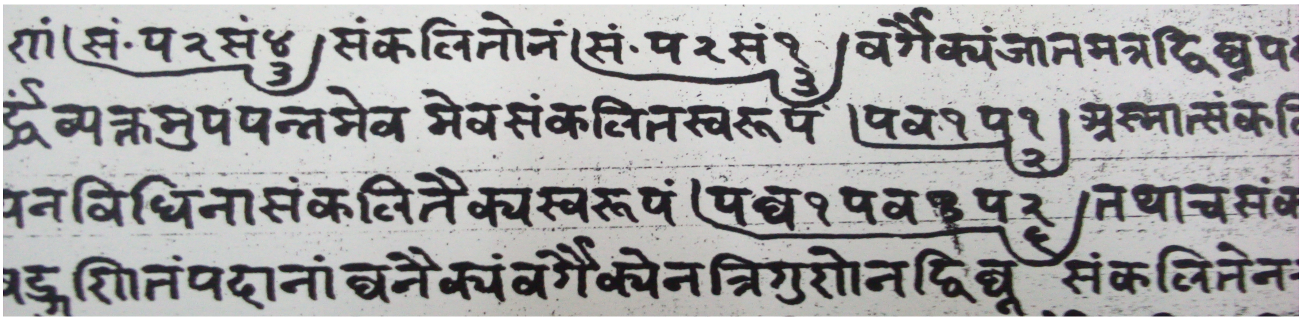


Fig. 5. Use of notation system in the *Nisṛṣṭārthadūtī*.

5.4. Sum of cubes of natural numbers

Moving on, Munīśvara also credits the same Rāmacandra for the following elegant proof[33] for the sum of cubes of integers. First, the text states the general principle that $S_n^2 - S_{n-1}^2 = n^3$, and then demonstrates an example:

> पदसङ्कलितम् एकोनपदसङ्कलितम् अनयोरन्तरं पदम् । तस्य घनः सङ्कलितवर्गान्तरम् । यथा पदं ३, सङ्कलितं ६, एकोनपदसङ्कलितं ३, अनयोर्वर्गौ साध्यौ । [...] एवं सर्वत्र सङ्कलितवर्गान्तरतुल्यः पदघन इति सिद्धम् ।
>
> The difference between summation of *pada* and summation of *pada*-minus-one is *pada*. Its cube is the difference of the squares of the summations. Example: *pada* 3, summation 6, summation of *pada*-minus-one terms 3, The squares of these to be obtained. [...] Similarly, everywhere it is established that the cube of *pada* is equal to the difference of squares of summations.

The opening sentence of the above quotation in mathematical representation essentially translates to

$$S_n - S_{n-1} = n. \tag{12}$$

Also, the relation

$$S_n + S_{n-1} = n^2, \tag{13}$$

was well known.[34] Taking the product of (12) and (13), we have

$$S_n^2 - S_{n-1}^2 = n^3. \tag{14}$$

This is what is stated in the second sentence of the above quote. Then Munīśvara proceeds to demonstrate the above equation with a numerical example. In the given example, when $n = 3$, we have $S_3 = 6$, and when $n = 2$, we have $S_2 = 3$. Therefore, we have $S_3^2 - S_2^2 = 27 = 3^3$.

As per the relation given in (14), the cube of every natural number n can be expressed as difference of the squares of the summations S_n and S_{n-1}. Using this relation, the text proceeds to prove the relation for the summation of cubes of natural numbers as follows:

$$\begin{aligned} S_{n^3} &= 1^3 + 2^3 + 3^3 + \cdots + n^3 \\ &= (S_1^2 - S_0^2) + (S_2^2 - S_1^2) + \\ &\qquad (S_3^2 - S_2^2) + \cdots + (S_n^2 - S_{n-1}^2) \\ &= S_n^2, \end{aligned}$$

as $S_0 = 0$. The simplicity and elegance of this proof is at once satisfying to a mathematician, as well as readily understood by a student.

6. REMARKS AND CONCLUSION

The proofs given in *Nisṛṣṭārthadūtī* that we have discussed in this paper are remarkable for a number of reasons. Firstly, they demonstrate how far short of the truth has been the critique of some scholars that Indian mathematics is bereft

[33] Munīśvara 1879, pp. 95–96.

[34] See the third proof in Section 5.1.

of proofs. This simplistic conclusion which has been drawn by merely looking at the source works and not the commentaries has resulted in perpetrating the false notion that Indian mathematicians did not discuss proofs of the mathematical results they employed. Secondly, the description of multiple proofs in the text also shows that Indian mathematicians were not merely satisfied with convincing themselves of the truth of a given mathematical result, but also enjoyed approaching a given problem from various angles, and satisfying themselves that all approaches yielded the same result. Thirdly, the use of Devanāgarī symbols in some places while explaining certain rules involving fairly complex algebraic manipulation provides clear evidence of the development of a nascent notation system, and indicates that Indian mathematicians were indeed performing symbolic manipulation. However, the extent to which this was used is not evident.

Moreover, the second proof in Section 5.3 demonstrates how visual representations can be weaved into mathematics to simplify complex problems into manageable parts. Since visual representations have a great appeal to young minds, such proofs will be impactful when taught to the students at high school level. Also, the proofs discussed in the text are as rigorous as one would expect from a modern mathematician. They are at once simple, elegant and intuitive, while being entirely free of any erroneous notions. Therefore, this text along with other texts such as *Gaṇita-yukti-bhāṣā* serves to remove misconceptions regarding lack of proofs in Indian mathematics. Finally, Munīśvara's attribution of certain proofs to Lakṣmīdāsa and Rāmacandra indicates the flow of ideas between scholars, as well as the existence of an academic honour code regarding attribution of ideas and concepts.

In light of the above, it is clear that the *Nisṛṣṭārthadūtī* is an important work in the pantheon of Indian mathematical texts. The study of this text reiterates the importance of studying commentaries of mathematical works, as they shine much light on the workings of the minds of mathematicians of that age, and the means they used to convince themselves regarding the truth of mathematical statements. The authors intend to present a more detailed account of the commentary *Nisṛṣṭārthadūtī* in a future publication to throw further light on this important text.

ACKNOWLEDGEMENTS

The authors would like to place on record their sincere gratitude to MHRD for the generous support extended to them to carry out research activities on Indian science and technology by way of initiating the Science and Heritage Initiative (SandHI) at IIT Bombay. The authors would also like to express their unreserved gratitude to the Sampurnanand Sanskrit University, Vārāṇasī for making available the manuscripts of *Nisṛṣṭārthadūtī*, which made this study possible.

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